GR Problem Set 3

Birkhoff Theorem and Neutron Stars

$c = 1$

$$-d\tau^2 = -B(r,t)\,dt^2 + A(r,t)\,dr^2 + r^2 d\Omega^2,$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

1a) Compute $\Gamma^{\mu}_{\lambda\sigma}$.

$$\mathcal{L} = -B\dot{t}^2 + A\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} \Rightarrow$$

$$\ddot{t} + \frac{B'_t}{B}\dot{t}^2 + \underbrace{\frac{B'_r}{B}\dot{r}\dot{t}}_{2\Gamma^{t}_{rt}\dot{r}\dot{t}} = \frac{1}{2}\frac{B'_t}{B}\dot{t}^2 - \frac{1}{2}\frac{A'_t}{B}\dot{r}^2$$

Compare with

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\lambda\sigma}\dot{x}^{\lambda}\dot{x}^{\sigma} = 0 \,!$$

$$\Rightarrow \boxed{\Gamma^{t}_{tt} = B'_t/2B} \quad \boxed{\Gamma^{t}_{rt} = B'_r/2B}$$

$$\boxed{\Gamma^{t}_{rr} = A'_{t}/2B}$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}$$

$$\frac{d}{d\lambda}\left(2A\dot{r}\right) = -B'_{r}\dot{t}^{2} + A'_{r}\dot{r}^{2} + 2r\dot{\theta}^{2} +$$

$$+ 2r\sin^{2}\theta\,\dot{\varphi}^{2}$$

$$2A\ddot{r} + 2A'_{r}\dot{r}^{2} + 2A'_{t}\dot{r}\dot{t} =$$

$$= -B'_{r}\dot{t}^{2} + A'_{r}\dot{r}^{2} + 2r\dot{\theta}^{2} +$$

$$+ 2r\sin^{2}\theta\,\dot{\varphi}^{2}$$

$$\Rightarrow \boxed{\Gamma^{r}_{rt} = A'_{t}/2A}$$

All other connection coefficients can be computed this way.

1b) $|g| = A(r,t)B(r,t)r^4\sin^2\theta$ ③

$$R_{tr} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial t\,\partial r} - \frac{\partial \Gamma^\lambda_{rt}}{\partial x^\lambda} + \Gamma^\eta_{r\lambda}\Gamma^\lambda_{t\eta} -$$

$$- \frac{1}{2}\Gamma^\eta_{rt}\frac{\partial \ln|g|}{\partial x^\eta}$$

$$\ln|g| = \ln A + \ln B + \ln r^4\sin^2\theta$$

$$\partial_t \ln|g| = \frac{A'_t}{A} + \frac{B'_t}{B}$$

$$\partial^2_{rt}\ln|g| = \partial_r\left(\frac{A'_t}{A} + \frac{B'_t}{B}\right)$$

Now, $\Gamma^\lambda_{rt} \neq 0$ for $\lambda = r$ or t only

(show this!)

Then

$$\frac{1}{2}\frac{\partial^2 \ln|g|}{\partial t\,\partial r} - \partial_r\Gamma^r_{rt} - \partial_t\Gamma^t_{rt}\text{, where}$$

$$\Gamma^r_{rt} = A'_t/2A \qquad \Gamma^t_{rt} = B'_r/2B$$

$$\frac{1}{2}\frac{\partial^2 \ln|g|}{\partial t\,\partial r} = \frac{1}{2}\left[\frac{A''_{rt}}{A} + \frac{B''_{rt}}{B} - \frac{A'_t A'_r}{A^2} - \frac{B'_t B'_r}{B^2}\right]$$

$$\partial_r \Gamma^r_{rt} + \partial_t \Gamma^t_{rt} = \frac{1}{2}\left[\frac{A''_{tr}}{A} - \frac{A'_t A'_r}{A^2} + \frac{B''_{tr}}{B} - \frac{B'_r B'_t}{B^2}\right]$$

$$\Rightarrow \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial t\,\partial r} - \partial_r \Gamma^r_{tr} - \partial_t \Gamma^t_{rt} = 0.$$

1c) $\Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\varkappa\eta} - \frac{1}{2}\Gamma^\eta_{\mu\varkappa}\partial_\eta \ln|g|$ - ?

$\Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\varkappa\eta}$ with $\mu = t$, $\varkappa = r$:

$$\Gamma^\eta_{t\lambda}\Gamma^\lambda_{r\eta} = \Gamma^t_{tt}\Gamma^t_{rt} + \Gamma^r_{tt}\Gamma^t_{rr} + \Gamma^r_{tr}\Gamma^r_{rr} +$$

$$+ \Gamma^t_{tr}\Gamma^r_{rt} = \frac{B'_t B'_r}{4B^2} + \frac{A'_t A'_r}{4A^2} + \frac{B'_r A'_t}{4AB}\cdot 2$$

On the other hand:

$$-\frac{1}{2}\Gamma^{\eta}_{tr}\,\partial_{\eta}\ln ABr^4\sin^2\theta =$$

$$= -\frac{1}{2}\Gamma^{r}_{tr}\,\partial_{r}\left(\ln A+\ln B+4\ln r\right)-$$

$$-\frac{1}{2}\Gamma^{t}_{tr}\,\partial_{t}\left(\ln A+\ln B\right) =$$

$$= -\frac{A'_t}{4A}\left(\frac{A'_r}{A}+\frac{B'_r}{B}+\frac{4}{r}\right)-$$

$$-\frac{B'_r}{4B}\left(\frac{A'_t}{A}+\frac{B'_t}{B}\right)$$

$$\Rightarrow \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\varkappa\eta}-\frac{1}{2}\Gamma^{\eta}_{\mu\varkappa}\partial_{\eta}\ln|g| = -\frac{A'_t}{rA}$$

$$\Rightarrow R_{tr} = -A'_t/rA$$

Es eqs: $R_{\mu\nu}=0 \Rightarrow A'_t = 0.$

2. Oppenheimer - Volkoff - Tolman eq.

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right),$$

where $T = T^\mu{}_\mu$.

$$T_{\mu\nu} = P(r)\, g_{\mu\nu} + (\rho + P)\, u_\mu u_\nu$$

$u^i = 0$
$i = 1, 2, 3.$

Ansatz (spher. symm., static):

$$ds^2 = -B\, dt^2 + A\, dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

e.o.m. $\Rightarrow$ $\left(\frac{r}{A}\right)' = 1 - \frac{8\pi G \rho r^2}{c^2}$

$\oplus$ $A(0)$ - finite $\Rightarrow A(r) = \left(1 - \frac{2G\mathcal{M}(r)}{r}\right)^{-1}$,

where $\mathcal{M}(r) = 4\pi \int_0^r r^2 \rho(r)\, dr$

Also, $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow$

$$B'/B = -2P'/(\rho c^2 + P) \Rightarrow$$

$$\frac{dP}{dr} = -\left(\frac{G\mathcal{M}(r)}{r^2} + \frac{4\pi G P r}{c^2}\right)\left(\rho + P/c^2\right) \times$$

$$\times \left(1 - \frac{2G\mathcal{M}(r)}{c^2 r}\right)^{-1}$$

With e.o.s. $P = P(\rho)$.

$\Rightarrow$ system of eqs. for $\rho(r)$.

2a) $\mathcal{F} = M - \lambda N$ $\qquad \delta \mathcal{F} = 0 \Rightarrow$ OVT eq.

$$\mathcal{F} = \int_0^\infty 4\pi r^2 \rho(r)\, dr - \lambda \int_0^\infty 4\pi n(r) r^2 A^{1/2} dr$$

A note about M:

$$M = \int_0^\infty 4\pi r^2 \rho(r)\, dr = \int_0^R 4\pi r^2 \rho(r)\, dr =$$

$$= \int_0^R (\rho + 3P)\sqrt{AB}\, 4\pi r^2 dr, \text{ since}$$

$$\begin{cases} \mathcal{M}'(r) = 4\pi \rho(r) r^2 \\ P'(r) = \text{OVT eq} \end{cases}$$

$$\Rightarrow \int_0^R 4\pi \rho r^2 \sqrt{AB}\, dr = \int_0^R \mathcal{M}'(r) \sqrt{AB}\, dr =$$

$$= \mathcal{M}(r) \Big|_0^R - \int_0^R \mathcal{M}(r) \left(\sqrt{AB}\right)'_r dr$$

$$\mathcal{M}(0) = 0 \qquad \mathcal{M}(R) = M$$

Similarly for the term with P

$$M = \int_0^R (\rho + 3P)\, 4\pi r^2 A^{1/2}(r) B^{1/2}(r)\, dr$$

Non-rel: $\quad B^{1/2} = 1 + \Phi(r)/c^2 + \cdots$

(for $r > R$, $B^{1/2} = 1 - GM/c^2 r + \cdots$)

$\Rightarrow$ can show that

$$M = \int_0^R \rho \, dV + \frac{1}{2} \underbrace{\int_0^R \rho \, \Phi(r) \, dV}_{\text{grav. pot. energy of a ball}} + \cdots$$

$$dV = 4\pi r^2 A^{1/2} dr$$

See e.g. Tolman, §98 of „Relativity, Thermod. and Cosmology" and Press, Teukolsky ++ problems 16.23, 16.24.

2b) $\delta F / \delta \rho$ -?

$$\delta F = \delta \left[\int_0^\infty 4\pi r^2 \rho \, dr - \lambda \int_0^\infty 4\pi n r^2 A^{1/2} dr \right]$$

$$= \int_0^\infty 4\pi r^2 \delta\rho \, dr - \lambda \int_0^\infty 4\pi r^2 \delta n \, A^{1/2} dr -$$

$$- \lambda \int_0^\infty 4\pi n r^2 \, \delta (A^{1/2}) \, dr$$

$$A = \left(1 - 2G\mathcal{M}(r)/rc^2\right)^{-1},$$

$$\mathcal{M}(r) = \int_0^r 4\pi r^2 \rho(r)\, dr$$

$$\delta A^{1/2} = \frac{1}{2} A^{-1/2}\, \delta A$$

$$\delta A = A^2 \frac{2G}{rc^2}\, \delta\mathcal{M}$$

$$\Rightarrow \delta A^{1/2} = A^{3/2} \frac{G\,\delta\mathcal{M}}{rc^2}$$

$$\Rightarrow \delta F = \int_0^\infty 4\pi r^2 \left[\delta\rho - \lambda A^{1/2}\delta n - \lambda \frac{G A^{3/2}}{c^2 r} n\,\delta\mathcal{M}\right] dr$$

$\Rightarrow$ need to express δn and $\delta\mathcal{M}$ via $\delta\rho$.

2c) $dE = -PdV + \cancel{TdS}$

$$\delta\left(\rho c^2 V\right) = -P\,\delta V$$

If $nV = \text{const} \Rightarrow$

$$\delta n\, V + n\, \delta V = 0$$

$$\Rightarrow \quad \delta\rho c^2 V + \rho c^2 \delta V = P \frac{V}{n} \delta n$$

$$\Rightarrow \quad c^2 \delta\rho = \frac{P}{n} \delta n + \rho c^2 \frac{\delta n}{n}$$

$$\Rightarrow \boxed{\delta n = \frac{n\, \delta\rho}{\rho + P/c^2}}$$

2d) Since $\mathcal{M}(r) = \int_0^r 4\pi r'^2 \rho(r') dr'$,

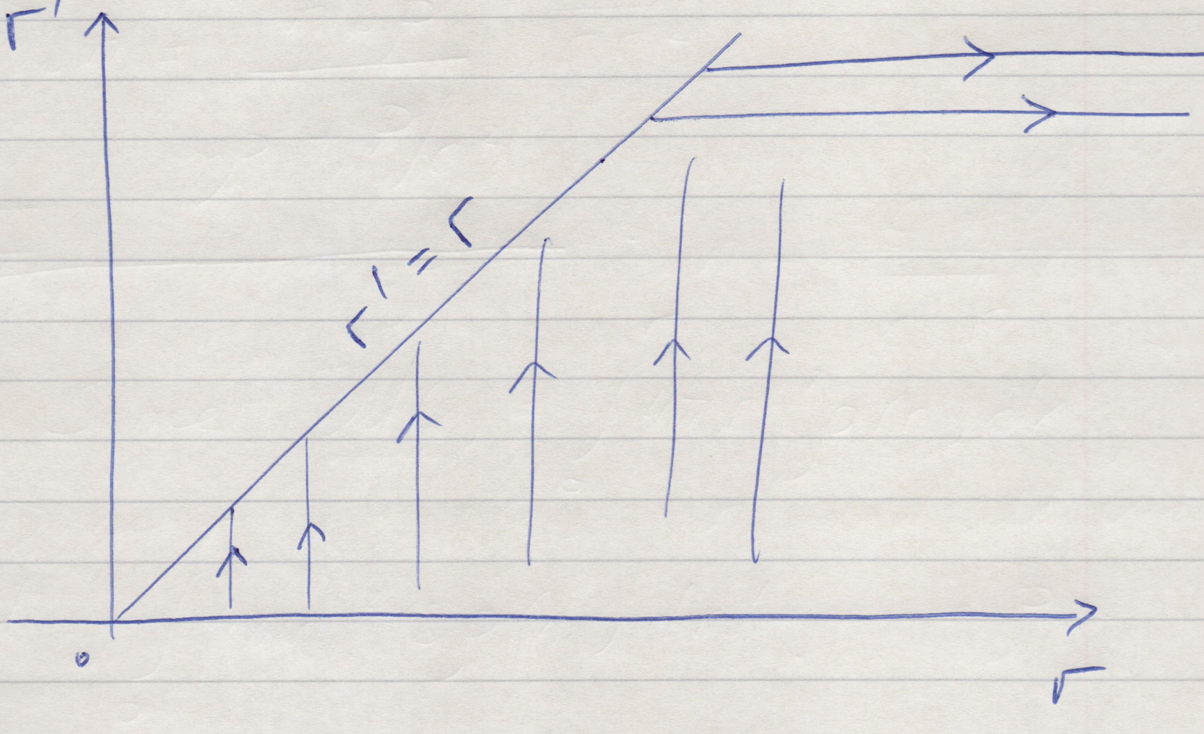
$$\delta \mathcal{M} = \int_0^r 4\pi r'^2 \delta\rho(r') dr'.$$

$$\delta \mathcal{F} = \int_0^\infty 4\pi r^2 \Big[\delta\rho - \lambda A^{1/2} \frac{n\, \delta\rho}{\rho + P/c^2} \; -$$

$$- \lambda \frac{G A^{3/2}}{c^2 r} n \left[\int_0^r 4\pi r'^2 \delta\rho(r') dr' \right] dr$$

Note that $\int_0^\infty dr \int_0^r dr' \, \delta\rho(r') =$

$$= \int_0^\infty dr' \int_{r'}^\infty dr \, \delta\rho(r') = \int_0^\infty dr \int_r^\infty dr' \, \delta\rho(r)$$



$$\delta \bar{F} = \int_0^\infty 4\pi r^2 \delta\rho \left[1 - \frac{\lambda A^{1/2} n}{\rho + P/c^2} - \frac{\lambda G}{c^2} \int_r^\infty 4\pi r' n A^{3/2} dr' \right] dr$$

2e) $\delta \mathcal{F} = 0$ implies

$$\frac{1}{\lambda} = \frac{n A^{1/2}}{\rho + P/c^2} + \frac{G}{c^2} \int_r^\infty 4\pi r' n A^{3/2} dr'$$

$\lambda = \text{const} \Rightarrow d/dr\,(1/\lambda) = 0$

$$\frac{d}{dr}\left(\frac{n A^{1/2}}{\rho + P/c^2}\right) = \frac{n' A^{1/2}}{\rho + P/c^2} - \frac{n A^{1/2}(\rho' + P'/c^2)}{(\rho + P/c^2)^2}$$

$$+ \frac{n}{\rho + P/c^2} \frac{d}{dr}\left(A^{1/2}\right).$$

$$A = \left(1 - \frac{2G\mathcal{M}(r)}{rc^2}\right)^{-1/2}, \quad \mathcal{M}(r) = \int_0^r 4\pi r^2 \rho(r) dr$$

$$\Rightarrow \frac{d}{dr} A^{1/2} = A^{3/2} \frac{G}{c^2}\left[4\pi \rho r - \frac{\mathcal{M}}{r^2}\right]$$

Also, $\delta n = n\,\delta\rho / (\rho + P/c^2)$

from 2c).

So,

$$\frac{n'}{\rho + P/c^2} - \frac{n(\rho' + P'/c^2)}{(\rho + P/c^2)^2} = -\frac{nP'/c^2}{(\rho + P/c^2)^2}$$

Combining all, we get

$$\frac{-nP'/c^2}{(\rho + P/c^2)^2} A^{1/2} + \frac{n}{\rho + P/c^2}\left[\frac{4\pi G\rho r}{c^2} - \frac{GM}{c^2 r^2}\right] A^{3/2}$$

$$-\frac{4\pi r G n A^{3/2}}{c^2} = 0$$

$\Rightarrow$

$$P' = -\frac{GM}{r^2}\left(1 + \frac{4\pi r^3 P}{Mc^2}\right) A \left(\rho + P/c^2\right)$$

which is the OVT eq.

Thus, OVT eq. can be derived from var. principle under certain assumptions.

Problem 3

3a) $$T^{\mu\nu} = P g^{\mu\nu} + (\rho + P/c^2) u^\mu u^\nu$$

$$\nabla_\mu T^{\mu\nu} = 0 \quad \text{or} \quad \nabla_\mu T^\mu{}_\nu = 0 :$$

$$\frac{\partial P}{\partial x^\nu} + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} \left[\sqrt{|g|} (\rho + P/c^2) u^\mu u_\nu \right] -$$

$$- \Gamma^\mu_{\nu\lambda} (\rho + P/c^2) u_\mu u^\lambda = 0$$

The term $-\Gamma^\mu_{\nu\lambda} u_\mu u^\lambda$ can be re-written

using $\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\varkappa} (\partial_\lambda g_{\varkappa\nu} + \partial_\nu g_{\varkappa\lambda} - \partial_\varkappa g_{\lambda\nu})$

$$-\Gamma^\mu_{\nu\lambda} u_\mu u^\lambda = -\frac{1}{2} u^\varkappa u^\lambda (\partial_\lambda g_{\varkappa\nu} + \partial_\nu g_{\varkappa\lambda} - \partial_\varkappa g_{\lambda\nu}) =$$

$$= -\frac{1}{2} u^\varkappa u^\lambda \partial_\nu g_{\varkappa\lambda}, \quad \text{since}$$

$$u^\varkappa u^\lambda (\partial_\lambda g_{\varkappa\nu} - \partial_\varkappa g_{\lambda\nu}) \equiv 0.$$

We have $u^{\varkappa} \partial_\nu g_{\varkappa\lambda} u^\lambda =$

$$= u^\varkappa u^\lambda \partial_\nu g_{\varkappa\lambda} + u^\varkappa g_{\varkappa\lambda} \partial_\nu u^\lambda$$

$$\Rightarrow \quad -\frac{1}{2} u^\varkappa u^\lambda \partial_\nu g_{\varkappa\lambda} = -\frac{1}{2}\left(u^\lambda \partial_\nu u_\lambda - u_\lambda \partial_\nu u^\lambda\right).$$

Since $u^\lambda u_\lambda = -c^2$, $u_\lambda \partial_\nu u^\lambda + u^\lambda \partial_\nu u_\lambda = 0$

$$\Rightarrow -\frac{1}{2} u^\varkappa u^\lambda \partial_\nu g_{\varkappa\lambda} = u_\lambda \partial_\nu u^\lambda$$

$$\Rightarrow -\Gamma^\mu_{\nu\lambda} u_\mu u^\lambda = u_\mu \partial_\nu u^\mu .$$

3b) $\Omega = d\phi/dt = \text{const}$

$$U^\phi = d\phi/d\tau = \Omega U^0, \quad U^0 = dt/d\tau$$

With $U^\mu = (U^0, 0, 0, U^\phi)$ the equation

$$\frac{\partial}{\partial x^\nu} P + \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu}\left[|g|^{1/2}(\rho + P/c^2) U^\mu U_\nu\right] +$$

$$+ (\rho + P/c^2) U_\mu \frac{\partial U^\mu}{\partial x^\nu} = 0 \quad \text{becomes}$$

$$\partial_\nu P + (\rho + P/c^2)\left[U_0\, \partial_\nu U^0 + U_\phi\, \partial_\nu U^\phi\right] = 0$$

(Note that ρ, P are indep. of t, ϕ)

or

$$\partial_\nu P + (\rho + P/c^2)\left[U_0 + \Omega U_\phi\right] \partial_\nu U^0 = 0$$

Since $U_0 U^0 + U_\phi U^\phi = -c^2$, we have

$$(U_0 + \Omega U_\phi)\, U^0 = -c^2 \quad \text{and}$$

$$\boxed{\partial_\nu P - (\rho c^2 + P)\, \partial_\nu \ln U^0 = 0.}$$

3c) If ρ and P are constants on the same surfaces given by eq. $F = \text{const}$ (e.g. $F = -c^2 t^2 + x_i^2 = \text{const}$), then $P = P(F(x))$ and $\rho = \rho(F(x))$.

Then $\partial_i P = \frac{dP}{dF}\partial_i F$ and $\partial_j \rho = \frac{d\rho}{dF}\partial_j F$

and $\partial_i P \partial_j \rho = \frac{dP}{dF}\frac{\partial \rho}{\partial F}\, \partial_i F \partial_j F =$

$= \partial_j P\, \partial_i \rho$.

3d) We have $\partial_\nu P = (\rho c^2 + P)\, \partial_\nu \ln U^0$

$\nu = \kappa$ (since nothing depends on t)

$\partial_\kappa P = (\rho c^2 + P)\, \partial_\kappa \ln U^0$

Computing curl: $(\mathrm{curl}\, \partial_\kappa P)_i = \overset{\varepsilon_{ijk}\partial_j \partial_\kappa P}{0} =$

$= \varepsilon_{ijk}\, \partial_j \left[(\rho c^2 + P)\, \partial_\kappa \ln U^0 \right] =$

$= \varepsilon_{ijk} \left(\partial_j \rho c^2 + \partial_j P \right) \underbrace{\partial_\kappa \ln U^0}_{\substack{\text{replace with} \\ \partial_\kappa P / (\rho c^2 + P)}} + \underbrace{\varepsilon_{ijk} (\rho c^2 + P)\, \partial_j \partial_\kappa \ln U^0}_{= 0}$

$\Rightarrow \varepsilon_{ijk}\, \partial_j \rho\, \partial_\kappa P = 0 \Rightarrow$ surfaces of const ρ and P coincide (see 3c).

Since $\partial_\kappa P \sim \partial_\kappa \ln U^0$, surfaces of const P and U^0 coincide.

$\Rightarrow$ surface clocks on the surface of const P (or ρ) run at the same rate. We only used ϕ independence

$\Rightarrow$ no <u>spherical</u> symm. assumed (Kerr OK).

<u>Note</u>: $\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

(terms with Γ cancel)

<u>Note</u>: surface of const U^0:

$$g_{\mu\nu} U^\mu U^\nu = -c^2$$

$$\Rightarrow (U^0)^2 (g_{tt} + 2 g_{t\phi} \Omega + g_{\phi\phi} \Omega^2) = \text{const}$$

Surface of the rot. star obeys

$$g_{tt} + 2 g_{t\phi} \Omega + g_{\phi\phi} \Omega^2 = \text{const.}$$

Problem 4

4a) This relies heavily on the details presented in Section 8.5 (Neutron stars) of the Lecturer's Notes.

The EOS is parametrised as

$$\begin{cases} \rho(t) = \dfrac{3\rho_{char}}{32}(\sinh t - t) \\ P(t) = \dfrac{\rho_{char} c^2}{32}\left[\sinh t - 8\sinh\dfrac{t}{2} + 3t\right], \end{cases}$$

where $t = 4 \operatorname{arcsinh} p_F / m_n c$,

$$\rho_{char} = \frac{8\pi}{3} \frac{m_n^4 c^3}{h^3} \simeq 6.11 \cdot 10^{18}\ kg/m^3.$$

This is EOS of relativ. <u>ideal</u> Fermi-gas.

Now, OVT eq is

$$\frac{dP}{dr} = -\left(\frac{GM}{r^2} + \frac{4\pi G P r}{c^2}\right)\left(\rho + \frac{P}{c^2}\right)\left(1 - \frac{2GM}{c^2 r}\right)^{-1},$$

where $\dfrac{dM}{dr} = 4\pi r^2 \rho(r)$.

There are also re-definitions

$$r \equiv c\,(G\rho_{char})^{-1/2}\, y$$

$$\mathcal{M} \equiv c^3 G^{-3/2} \rho_{char}^{-1/2}\, m$$

We can write OVT eq as

$$\frac{1}{\dot{P}}\frac{dP}{dr} = -\frac{1}{\dot{P}}\left(\frac{G\mathcal{M}}{r^2} + \frac{4\pi G P r}{c^2}\right)\left(\rho + \frac{P}{c^2}\right)\left(1 - \frac{2G\mathcal{M}}{c^2 r}\right)^{-1}$$

where $\dot{P} = dP/dt$, and then use y instead of r and m instead of $\mathcal{M}$:

$$\Rightarrow \frac{dt}{dy} = -c^2 \frac{m}{y^2} \frac{(\rho + P/c^2)}{\dot{P}}\left(1 + \frac{4\pi P r^3}{\mathcal{M} c^2}\right)\left(1 - \frac{2G\mathcal{M}}{c^2 r}\right)^{-1}$$

But

$$\frac{\rho + P/c^2}{\dot{P}} = \frac{4}{c^2} \frac{(\sinh t - 2\sinh t/2)}{\cosh t - 4\cosh t/2 + 3}$$

etc $\Rightarrow$ by direct substitution we obtain

$$\frac{dt}{dy} = -\frac{4m}{y^2(1-2m/y)}\left(\frac{\sinh t - 2\sinh t/2}{\cosh t - 4\cosh t/2 + 3}\right) \times$$

$$\times\left(1 + \frac{\pi y^3}{8m}\left[\sinh t - 8\sinh t/2 + 3t\right]\right)$$

and $$\frac{dm}{dy} = \frac{3\pi y^2}{8}(\sinh t - t)$$

At $y=0$ (i.e. $r=0$) we have $t = t_0$

$\Rightarrow m(y) = \frac{\pi y^3}{8}(\sinh t_0 - t_0)$ for small y.

4b) With $t \to \infty$, we have

$$\frac{dt}{dy} = -\frac{4m}{y^2(1-2m/y)}\frac{\sinh t}{\cosh t}\left(1 + \frac{\pi y^3}{8m}\sinh t\right)$$

+ corrections

Also, $\rho(t) = \frac{3\rho_{char}}{32}\sinh t$ + corrections

So, $$dp = \frac{3\rho_{char}}{32} \cosh t \, dt + \dots$$

$$\Rightarrow \frac{\cancel{32}\, dp}{\cancel{3\rho_{char}}\, \cancel{\cosh t}} = -\frac{4m}{y^2\left(1-\frac{2m}{y}\right)} \frac{\cancel{32}\,\rho}{\cancel{3\rho_{char}}} \times$$

$$\times \underbrace{\left(1 + \frac{\pi y^3}{8m}\frac{32\rho}{3\rho_{char}}\right)}_{1 + \frac{4\pi r^3 \rho}{3M}} \frac{1}{\cancel{\cosh t}}$$

$$\Rightarrow \frac{d\rho}{dr} = -\frac{GM}{r^2c^2} \frac{4\rho(r)}{1 - 2GM/rc^2}\left(1 + \frac{4\pi r^3 \rho}{3M(r)}\right)$$

This is OVT in the ultrarel. limit, when $p_F \gg m_n c$ or $E \gg mc^2$.