Problem Set 2. **General Relativity, HT21**

EQUILIBRIUM, FLOWS, AND ORBITS IN GENERAL RELATIVITY

1.) *Hydrostatic Equilibrium in GR.* Model a neutron star atmosphere with a simple equation of state: $P = K\rho^\gamma$, where P is pressure, ρ is mass density, γ is the adiabatic index and K is a constant. Assume that $g_{00} = -(1 - 2GM/rc^2)$, where M is the mass of the star and r is radius. If $\rho = \rho_0$ at the surface $r = R_0$, solve the equation of hydrostatic equilibrium to show that

$$\frac{1 + K\rho^{\gamma-1}/c^2}{1 + K\rho_0^{\gamma-1}/c^2} = \left(\frac{1 - R_S/r_0}{1 - R_S/r}\right)^\alpha$$

where $R_S = 2GM/c^2$ is the so-called Schwarzschild radius, and $2\alpha\gamma = \gamma - 1$. (Hint: See §4.6 of the notes.) What is the Newtonian limit of the above equation? Express your answer in terms of the speed of sound a, $a^2 = \gamma P/\rho$ and the potential $\Phi(r) = -GM/r$. (OPTIONAL: For those who have studied fluids, what quantity is being conserved in the Newtonian limit?)

2.) *Bondi Accretion: go with the flow.* To get some practise working with the equations of GR as well as some insight into relativistic dynamics in a practical problem in astrophysics, consider what is known as (relativistic) Bondi Accretion, the spherical flow of gas into a black hole. (The original Bondi accretion problem was Newtonian accretion onto an ordinary star.) We assume a Schwarzschild metric in the usual spherical coordinates:

$$g_{00} = -(1 - 2GM/rc^2),\ g_{rr} = (1 - 2GM/rc^2)^{-1},\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2\sin^2\theta.$$

2a.) First, let us assume that particles are neither created or destroyed. So particle number is conserved. If n is the particle number density in the local rest frame of the flow, then the particle flux is $J^\mu = nU^\mu$, where U^μ is the flow 4-velocity. Justify this statement, and using §4.5 in the notes, show that particle number conservation implies:

$$J^\mu{}_{;\mu} = 0.$$

If nothing depends upon time, show that this integrates to

$$nU^r|g'|^{1/2} = \text{constant},$$

where g' is the determinant of $g_{\mu\nu}$ divided by $\sin^2\theta$, and U^r is...well, you tell me what U^r is.

2b.) We move on to energy conservation, $T^{t\nu}{}_{;\nu} = 0$. (Refer to §4.6 in the notes.) Show that the only nonvanishing affine connection that we need to use is

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}\frac{\partial \ln|g_{tt}|}{\partial r}$$

Derive and solve the energy equation. Show that its solution may be written

$$(P + \rho c^2)U^r U_t |g'|^{1/2} = \text{constant}$$

where $U_t = g_{t\mu}U^\mu$, and ρ is the total energy density of the fluid in its rest frame, including any thermal energy.

2c.) We next define

$$\varpi = \mu n,$$

where μ is the rest mass per particle and ϖ is a Newtonian density. This is not to be confused with ρ, the true relativistic energy density divided by c^2. P and ϖ are assumed to be related by a simple power law relationship,

$$P = K\varpi^\gamma$$

where K is a constant, and γ is called the adiabatic index. This is not an entirely artificial problem: it is valid for cold classical particles ($\gamma = 5/3$) or hot relativistic particles ($\gamma = 4/3$). The first law of thermodynamics then tells us that the thermal energy per unit volume is

$$\epsilon = \frac{P}{\gamma - 1}$$

(You needn't derive that here, just use it!) Show that this implies:

$$\rho = \varpi + \frac{P}{c^2(\gamma - 1)}.$$

2d.) Verify that

$$|g'| = r^4$$

and using $g_{\mu\nu}U^\mu U^\nu = -c^2$, show that

$$U_t = \left[c^2 - \frac{2GM}{r} + (U^r)^2\right]^{1/2}$$

(Take care to distinguish U^t and U_t.)

2e.) With

$$a^2 = \gamma P/\varpi,$$

(this is the speed of sound in a nonrelativistic gas), combine our mass and energy conservation equations to show that

$$\left(\mathbf{c^2 + \frac{a^2}{\gamma - 1}}\right)^{\mathbf{2}} \left(\mathbf{c^2 + U^2 - \frac{2GM}{r}}\right) = \mathbf{constant}.$$

We have dropped the superscript r on U^r for greater clarity. How does a^2 depend upon ϖ? The other equation we shall use is just that of mass conservation itself. Show that this may be written as

$$4\pi\varpi r^2 U = \dot{m},$$

which defines the net, constant mass accretion rate $\dot{m} < 0$. With a^2 depending entirely on ϖ, and $\varpi = \dot{m}/(4\pi r^2 U)$, the equation in **boldface** becomes a single algebraic equation for U as a function of r, and the formal solution to our problem.

2f.) Three final simple tasks for now:

i) Show that the constant on the right of the **bold** equation of problem (2e) is

$$c^2\left(c^2 + \frac{a_\infty^2}{\gamma - 1}\right)^2$$

where a_∞ is the sound speed at infinite distance from the black hole, if the gas starts accreting from rest.

ii) Show that the Newtonian limit of the equation is

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1}$$

where v is the ordinary velocity, not the 4-velocity. This is a statement that a quantity known as enthapy (energy plus the work done by pressure) is conserved. This is the original nonrelativistic Bondi 1952 solution for accretion onto a star.

iii) Show that as r approaches the Schwarzschild radius $R_S = 2GM/c^2$, then if $a \ll c$ everywhere, then dr/dt satisfies the condition of a "null geodesic," a fancy way to say the inflow follows the equation of light:

$$\frac{dr}{dt} = -c(1 - R_S/r).$$

Like stalled photons, from the point of view of a distant observer, the flow never crosses R_S.

3a.) *Kinematic and gravitational redshifts.* One of the most important observational black hole diagnostics is a calculation of the radiation spectrum from the surrounding disc. In particular we are interested in how the frequency of a photon is shifted due to space-time distortions and relativistic kinematics. Show that:

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu(R)V^\mu(R)}{p_\mu(E)V^\mu(E)}$$

where R denotes the received the photon and E the emitted photon, ν is a frequency (not an index here!), p_μ a covariant photon 4-momentum, and V^μ is the normalised 4-velocity in the form $(dt/d\tau, d\mathbf{x}/cd\tau)$ for the emitted material (E) or the distant observer at rest (R).

3b.) In the problem at hand, the observer views the disc edge-on, in the plane of the disc. The gas moves in circular orbits

$\odot$————→ observer $\gtrdot$

Show that in t, r, θ, ϕ coordinates for the $0, 1, 2, 3$ components,

$$V^\mu(R) = (1, 0, 0, 0), \quad V^\mu(E) = V_E^0(1, 0, 0, d\phi/cdt), \text{ with } V_E^0 = dt/d\tau$$

Then, using $g_{\mu\rho}V^\mu V^\rho = -1$), conclude that

$$V_E^0 = (1 - 3GM/rc^2)^{-1/2}.$$

You may use a result from problem (5c) below. (You will prove it later!)

3c.) Finally, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} \left(1 + \frac{\Omega p_\phi(E)}{cp_0(E)}\right)^{-1}, \quad \Omega^2 = GM/r^3.$$

A result of problem (3) from Problem Set 1 may be useful.

From disk material moving at right angles across the line of sight, ν_R/ν_E reduces to

$$(1 - 3GM/rc^2)^{1/2}.$$

Why? From disk material moving precisely along the line of sight, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - 3GM/rc^2\right)^{1/2} / \left(1 \pm (rc^2/GM - 2)^{-1/2}\right)$$

(Hint: $g^{\nu\rho}p_\nu p_\rho = 0$.) Interpret the $\pm$ sign. In general, the photon paths must be calculated from the dynamical equations to determine the $p(E)$ ratio.

4a.) *The perihelion advance of Mercury.* In the notes we found that the differential equation for $u = 1/r$ for Mercury's orbit could be written as follows. $u = u_N + \delta u$ with the Newtonian solution u_N given by

$$u_N = (GM/J^2)(1 + \epsilon\cos\phi)$$

and the differential equation for δu is

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3(GM)^3}{c^2J^4}(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi).$$

Show that this is equivalent to solving the real part of the equation

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a(b + 2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2)$$

where $a = 3(GM)^3/(c^2J^4)$ and $b = 1 + \epsilon^2/2$.

To solve this, try a solution of the form

$$\delta u = A_0 + A_1\phi e^{i\phi} + A_2 e^{2i\phi}$$

where the A's are constants. Why do we need an additional factor of ϕ in the A_1 term?

4b.) Show that the solution for $u = u_N + \delta u$ is

$$u = \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon\cos\phi + \epsilon a\phi\sin\phi$$

Since a is very small, show that this equivalent to

$$u = ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}[1 + \epsilon(\cos\phi(1-\alpha))]$$

where

$$\alpha = aJ^2/GM = 3(GM/Jc)^2$$

4c.) In the equation for u, the first two terms in a cause tiny (and unmeasurable) distortions in the shape of the ellipse, but do not affect the 2π perodicity in ϕ of the orbit. Show however that the final term, proportional to GM/J^2, results in a periastron advance of

$$\Delta\phi = 6\pi\left(\frac{GM}{cJ}\right)^2$$

each orbit. This is the classic Einstein result.

5a.) *Black hole orbits.* In Newtonian theory, the energy equation for a test particle in orbit around a point mass is

$$\frac{v^2}{2} + \frac{l^2}{2r^2} - \frac{GM}{r} = \mathcal{E}$$

where r is radius, v is the radial velocity, l the angular momentum per unit mass, $\mathcal{E}$ the constant energy per unit mass, and $-GM/r$ is of course the potential energy. For the Schwarzschild solution show that the integrated geodesic equation may also be written in the form

$$\frac{v_S^2}{2} + \frac{l_S^2}{2r^2} + \Phi_S(r) = \mathcal{E}_S$$

where r is the standard radial coordinate, l_S and $\mathcal{E}_S$ are constants, $\Phi_S(r)$ is an effective potential function, and $v_S = dr/d\tau$. Determine l_S and $\mathcal{E}_S$ in terms of the fundamental angular momentum and energy constants J and E from lecture (or the notes). Express $\Phi_S(r)$ in terms of l_S, $\mathcal{E}_S$, the speed of light c, GM and r. The form of l_S, $\mathcal{E}_S$, and Φ_S should be chosen to go over to their Newtonian counterparts in the limit $E \to c^2$, $c \to \infty$, $E - c^2 \to$ finite.

5b.) Sketch the effective potential $l_S^2/2r^2 + \Phi_S(r)$. Prove that there is always a potential minimum in Newtonian theory, but that this is not the case in general relativity. What is the mathematical condition for the existence of a potential minimum for Φ_S, and what does it mean physically if it does not exist?

5c.) Show that for the Schwarzschild metric, circular orbits satisfy

$$\Omega^2 = \frac{GM}{r^3},$$

exactly the Newtonian form. Here $\Omega \equiv d\phi/dt$ at the coordinate location r, where dt is the proper time interval at infinity. Derive expressions for E and J in terms of GM, c^2 and r.

5d.) Below what value of r does Φ_S not have any local extrema? (Answer: $6GM/c^2$.)