Problem Set 5. **General Relativity, HT24**

Working with the FRW metric

RECALL:

Dynamical evolution equation:

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = 2E \text{ (Energy Form)} = -\frac{c^2}{a^2} \text{ (Curvature Form)} = -kc^2 \text{ (FRW Form)}$$

FRW metric, R dimensions of length, $k = 0, \pm 1$:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1 - kr^2} + R^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Curvature form, with $R_0 = 1$ and R dimensionless, a^2 positive or negative:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1 - r^2/a^2} + R^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

1a.) *A big bang, but in empty space you say. Really?* Show that the dynamical field equation for the scale factor $R(t)$ for an *empty space* $\rho = 0$ leads to an FRW metric of the form

$$-d\tau^2 = -dt^2 + \frac{t^2 dr^2}{1 + r^2} + r^2 t^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Use $c = 1$ for this problem!

1b.) Wait...Surely empty space must be Minkowski spacetime. Though this metric does not look static, there *must* be a coordinate transformation that turns this metric into a static Minkowski form. In other words, we ought to be able to find two functions, s and T,

$$s = s(r,t), \quad T = T(r,t) \qquad \text{or equivalently} \qquad r = r(s,T), \quad t = t(s,T)$$

that transform the metric of part (1a) into an old friend:

$$-d\tau^2 = -dT^2 + ds^2 + s^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

By inspection, we must have

$$s(r,t) = rt.$$

Why "by inspection?" Explain convincingly why it is as simple as this, in just one to two sentences.

1c.) Using $s = rt$, and by then demanding that the coefficient of dT^2 be -1 after the coordinate change, show that $T = \sqrt{s^2 + t^2}$ (up to an additive function of s which you may safely discard), and thereby derive the second coordinate transformation:

$$T = t\sqrt{1 + r^2}.$$

Give the explicit functional forms for $r(s,T)$ and $t(s,T)$.

1d.) Complete the full coordinate transformation for $d\tau^2$ and verify in detail that the Minkowski metric emerges. You may find it to your advantage to express $\partial t/\partial s$ and $\partial r/\partial s$ in terms of r and t, and $\partial r/\partial T$ in terms of $\partial t/\partial T$, before you begin. This is a valuable lesson: it is easy to be fooled by coordinates.

2.) *A radiation/matter universe. Fire and brimstone!* Solve the dynamical cosmological equation (Energy Form) for $R(t)$ for the case of an arbitrary mixture of radiation and non-relativistic matter in a spatially flat universe ($E = 0$). Assume a current energy density of $\rho_{\gamma 0}c^2$, and a matter density ρ_{m0}. In terms of the "inferno ratio" $I = \rho_{\gamma 0}/\rho_{m0}$, you should find

$$(R+I)^{3/2} - 3I(R+I)^{1/2} + 2I^{3/2} = \frac{3\Omega_{m0}^{1/2} H_0 t}{2}$$

(Note: This cubic equation is simple enough that the analytic solution is useful. Here it is [no need to prove]:

$$\frac{R}{I} = 4\cos^2\left[\frac{1}{3}\cos^{-1} Q\right] - 1 = 1 + \cos\left(\frac{2}{3}\cos^{-1} Q\right), \quad Q = \frac{3H_0 t\Omega_{m0}^{1/2}}{4I^{3/2}} - 1$$

This holds as long as $-1 \leq Q < 1$. When $Q \geq 1$, replace cos and $\cos^{-1}$ with cosh and $\cosh^{-1}$.)

3.) *A bullet in an E-dS universe.* Shoot a bullet into an Einstein-de Sitter universe at start of time. Nothing is actually pushing or pulling the bullet, but each comoving observer will see the bullet fly by at a different velocity as it passes. The question is, how far does the bullet get? More precisely, what is the largest comoving coordinate distance r the bullet attains if it starts at $r = 0$, $R = 0$? The metric is standard E-dS:

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2 dr^2 + R^2 r^2 d\Omega^2$$

$R(t)$ is the usual scale factor. We will use $d\varpi = R dr$ for the proper physical distance. Other standard notation and results for reference: t_0 is the current age of the universe, $R = (t/t_0)^{2/3}$ for E-dS, $H_0 \equiv \dot{R}_0$.

3a.) The quantity $d\varpi/dt$ measures the bullet's velocity relative to expanding, comoving observers who are all moving away. Show that if the bullet has a measured velocity V_1 at some instant when it passes one such observer, then when the bullet overtakes another observer, a tiny distance $d\varpi$ farther away, the velocity V_2 this observer measures is

$$V_2 = V_1 - \frac{\dot{R}\, d\varpi}{R}\left(1 - \frac{V_1^2}{c^2}\right)$$

to first order in $d\varpi$. (You will need the special relativity velocity addition formula and a local Hubble's law. Full special relativity works locally because $d\varpi$ is a tiny distance, and in this tiny, *comoving* frame special relativity holds. The relativity bit only matters when V_1 is comparable to c, but we allow for that!) From this equation, show that the rate at which the measured $V = d\varpi/dt$ is changing with cosmic time is given by the differential equation

$$\frac{\dot{V}}{V(1 - V^2/c^2)} = -\frac{\dot{R}}{R}$$

where $\dot{V} = (V_2 - V_1)/dt = dV/dt$. Solve this equation and show that with $V = V_0$ at $t = t_0$, the solution is

$$\frac{V}{\sqrt{1 - V^2/c^2}} = \frac{U_0}{R}$$

where U_0 is the spatial component of the bullet 4-velocity corresponding to V_0 at time t_0. (N.B.: In this problem, subscript 0 will always denote "current time," not the 4-vector time-like component.)

3b.) The result of (3a.) shows that the product $\mathcal{P}R$ is constant, where $\mathcal{P}$ is the spatial component of the bullet 4-momentum. Show that, in this form, this is equivalent to an adiabatic expansion, either of photons (extreme relativistic particles), or classical particles (classical nonrelativistic gas). [Cosmic adiabatic expansion for photons correponds to the temperature T obeying $TR \sim constant$, while for a classical gas, adiabatic behaviour is $T\rho^{-2/3} \sim constant$, where ρ is the mass (or in this case number) density.] In other words, a gas of bullets would "cool" like an ordinary gas!

3c.) Solve the equation $d\varpi/dt = V(R)$ for the comoving coordinate r in an E-dS universe to obtain for our problem:

$$r(R) = \frac{c}{H_0} \int_0^R \frac{dx}{[x + c^2 x^3/U_0^2]^{1/2}}$$

and show therefore that as $R \to \infty$, the comoving coordinate $r \to r_{max}$, where

$$r_{max} = \frac{3.708\sqrt{U_0 c}}{H_0}$$

The numerical factor is

$$3.708 = \int_0^\infty \frac{dy}{(y + y^3)^{1/2}}$$

Even after an infinite amount of time, and even though this universe is decelerating, a fired bullet only reaches a finite value of comoving coordinate r for any finite U_0. But the bullet can reach arbitrarily large r, if V_0 approaches the speed of light.

4.) *Schwarzschild and FRW geometries.* How long does it take a classical matter dominated closed universe to collapse, starting at its maximum extent? Express your answer two ways: in terms of the current value of the density ρ_0 and Ω_{m0}, and then in terms of the density at maximum extent ρ_m. Now, suppose we take all the mass in a small sphere of radius r_0 with density ρ_m (the sphere is small so that we don't have to worry about non-Euclidian curvature: the mass is just $4\pi r_0^3 \rho_m/3$), and turn the matter into a Schwarzschild black hole. Calculate the *proper time* for a test particle to fall into the hole from radial coordinate r_0 in a Schwarzschild geometry. You should find exactly the same answer for the universe as a whole. (Sections 6.5 and 10.5 in the notes will be useful.) Can you account for this amazing agreement in a simple way?

5a.) *There and back again: a photon's tale.* For a closed, matter-dominated universe with current mass density ρ_0, show that

$$H_0^2(\Omega_{M0} - 1) = c^2/a^2$$

where

$$\Omega_{M0} = \frac{8\pi G\rho_0}{3H_0^2}$$

5b.) Consider the path of a photon (null geodesic) through this universe. With η defined in §10.5 in the notes:

$$R = \frac{1 - \cos\eta}{2(1 - \Omega_{M0}^{-1})}$$

show that the comoving photon coordinate satisfies

$$r = a\sin\eta$$

Describe the path of a photon through this universe if launched at $r = 0$ at $R = t = 0$. (Hint: Is the photon at the same location at $\eta = 0$ and $\eta = \pi$?)