Gravitational Radiation

1a.) *The Burke-Thorne Potential.* Consider the following unusual Newtonian potential, due to Burke and Thorne:

$$\Phi = \frac{GJ^{(5)}_{ij}}{5c^7} x^i x^j$$

where $J^{(5)}_{ij}$ is the traceless (energy) moment of inertia tensor, differentiated five times with respect to time:

$$J^{(5)}_{ij} = c^2 \frac{d^5}{dt^5}\left[\int \rho(x_i x_j - \delta_{ij} r^2/3)\, dV\right] = \frac{d^5}{dt^5}\left[I_{ij} - \delta_{ij} I_{kk}/3\right].$$

Here, $I_{ij} = c^2 \int \rho x_i x_j \, dV$ is the standard moment of inertia tensor. The indices i and j represent spatial Cartesian coordinates, and we use the Minkowski metric, so spatial index placement is unimportant. The radius $r^2 = x^i x^i$. Show that this potential gives rise to a force, $-\partial_i \Phi$, which is exactly analogous to the "radiation reaction force" in electromagnetism. In other words, show that we recover Einstein's gravitational energy loss formula,

$$\frac{dE}{dt} = \left\langle -\int \rho v_i \partial_i \Phi \, dV \right\rangle = -\frac{G}{5c^9}\langle \dddot{J}_{ij} \dddot{J}_{ij} \rangle.$$

This equation states that the work done by the force, averaged over time (this is the meaning of the angle brackets $\langle\rangle$) equals the rate at which energy is lost from the system. This also works for angular momentum loss as well. Show that:

$$\frac{dL}{dt} = -\left\langle \int \epsilon^{ijk} \rho x_i \partial_j \Phi \, dV \right\rangle = -\frac{2G}{5c^9}\left\langle \epsilon^{imk} \dddot{J}_{mn} \ddot{J}_{in} \right\rangle,$$

which states that the effect of "$\mathbf{r} \times \mathbf{F}$" torque, averaged over time, equals the angular momentum loss.

Here are some hints:

i.) When in doubt, integrate by parts, either in time or in space. The system is assumed to be spatially finite so that "integrals at infinity" may be ignored. Time averages over exact time derivatives may also be neglected. (Assumes periodic or strictly bounded motion.)

ii.) The equation of mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x^i} = 0$$

is used for the energy loss formula derivation. (You don't have to take the time here to prove this, though seriously, you ought to be familiar with mass conservation by now.)

iii.) You should find that for the angular momentum loss formula, the result holds either for the traceless moment of inertia J_{ij} or for I_{ij}.

2.) *Desert island GR.* Here we will construct a linear, weak field theory gravity from scratch. Then we will construct GR from scratch! (Well, practically.)

Imagine that it is 1912. Minkowski has formulated the concept of his spacetime geometry (1908), but has departed the scene. Einstein has had his happy (1907) Equivalence Principle thought, and has at last understood that gravity is a Riemannian geometric theory of a distorted Minkowski spacetime, and that the name of the game is to relate the derivatives of $g_{\mu\nu}$ to $T_{\mu\nu}$. But he knows nothing more. Let's help him out.

2a.) Our weak gravity field equation will need, on the left side, a sum of second derivatives of $g_{\mu\nu}$. More conveniently, we use derivatives of the small quantity $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Not only is the background spacetime geometry flat Minkowski, our coordinates are very close to Cartesian. So, with $h \equiv h^{\rho}{}_{\rho}$, there are but five combinations that could possibly appear:

$$\Box h_{\mu\nu},\ \partial_\mu\partial_\nu h,\ (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu),\ \eta_{\mu\nu}\Box h,\ \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda}$$

($\Box \equiv \partial^\rho\partial_\rho$. We use the handy notation $\partial^\mu = \partial/\partial x_\mu$, $\partial_\mu = \partial/\partial x^\mu$, and raise and lower indices on $h_{\mu\nu}$ with $\eta^{\rho\mu}$.) Justify this statement and explain fully.

2b.) We accordingly search for an equation of the form:

$$\Box h_{\mu\nu} + \alpha(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \beta\partial_\mu\partial_\nu h + \eta_{\mu\nu}(\gamma\Box h + \delta\partial_\rho\partial_\lambda h^{\rho\lambda}) = CT_{\mu\nu}$$

where $\alpha, \beta, \gamma, \delta$ and C are constants to be determined. You remember, of course, the stress tensor $T_{\mu\nu}$, now in Newtonian guise. We demand that $\partial^\mu T_{\mu\nu} = 0$ as an identity. What is the reason for this? Show that $\alpha = -1, \delta = 1, \gamma = -\beta$ follow:

$$\Box h_{\mu\nu} - (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \beta\partial_\mu\partial_\nu h - \eta_{\mu\nu}(\beta\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}) = CT_{\mu\nu}$$

2c.) By taking the trace of this last equation and using $T_{00} \gg T_{ii}$ (valid in the Newtonian limit — why?), show that

$$\partial_\rho\partial_\lambda h^{\rho\lambda} = \frac{3\beta - 1}{2}\Box h - \frac{CT_{00}}{2}$$

Be careful with signs and up-down indices.

2d.) Taking the static Newtonian limit of the (2b) final equation, show that

$$\nabla^2 h_{00} + \frac{1-\beta}{2}\nabla^2 h = \frac{C}{2}T_{00}$$

where ∇^2 is the usual Laplacian operator. Explain why this implies $\beta = 1$ and $C = -16\pi G$:

$$\Box h_{\mu\nu} - (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}) = -16\pi G T_{\mu\nu}$$

Compare this with section (9.1) in the notes and comment.

2e.) Given that the Ricci tensor $R_{\mu\nu}$ and $g_{\mu\nu}R^{\rho}{}_{\rho}$ are the only second rank tensors that are linear in the second derivatives of the metric tensor $g_{\mu\nu}$ when the curvature is weak, explain why the general field equations must take the form

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = -8\pi G T_{\mu\nu}$$

where $R \equiv R^\rho_\rho$. Notice: not a Bianchi identity in sight. If Einstein could only have seen this in 1912.

3a.) *Coordinate sinuosities, the speed of gravitational radiation, and the harmonic gauge.* Recall the linear fully covariant curvature tensor:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right).$$

For a plane wave of the form $h_{\mu\nu} = A_{\mu\nu}\exp(ik_\rho x^\rho)$ travelling in vacuum, show that

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(-k_\kappa k_\mu h_{\lambda\nu} + k_\kappa k_\lambda h_{\mu\nu} + k_\mu k_\nu h_{\lambda\kappa} - k_\nu k_\lambda h_{\mu\kappa}\right)$$

and that the linear vacuum field equation is

$$k_\kappa k^\rho \bar{h}_{\rho\mu} + k_\mu k^\rho \bar{h}_{\rho\kappa} - k^2 h_{\mu\kappa} = 0$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2$ and $k^2 = k^\rho k_\rho$. We do *not* yet assume that $k^2 = 0$, but shall try to deduce this.

3b.) Show that if $k^2 \neq 0$ then $R_{\lambda\mu\nu\kappa} = 0$. Yikes! No curvature. *A mere coordinate sinuosity propagating at the speed of thought.*

3c.) Finally, show that if we consider only disturbances propagating at the speed of light, then we must have $k^\rho \bar{h}_{\rho\sigma} = 0$. In other words, the harmonic gauge condition *must* be satisfied. You want gravitational radiation to travel at the speed of light and to actually produce curvature? No choice: use a harmonic gauge.

4.) *Radiation from a parabolic fly by.* The Peters—Mathews formula for the time-averaged gravitational wave luminosity of a binary system in an elliptical orbit (with semi-major axis a, masses m_1 and m_2, $M \equiv m_1 + m_2$, eccentricity ϵ) is given by (c is now back in the equation):

$$\langle L_{GW}\rangle = \frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2 M}{a^5}\left[\frac{1 + (73/24)\epsilon^2 + (37/96)\epsilon^4}{(1-\epsilon^2)^{7/2}}\right]$$

It's derivation is outlined in the notes (§9.6), or you may take it on perfect good faith from your humble instructor, however startling it may seem. Using this result, show that the total gravitational wave energy emitted by a single parabolic encounter between two bodies is

$$E_{GW} = \frac{85\pi\sqrt{2}}{24}\frac{G^{7/2}M^{1/2}m_1^2 m_2^2}{c^5 b^{7/2}}$$

where b is radius of closest approach. Recall that for a parabolic orbit, the radius r and aximuth ϕ are related by $r(1+\cos\phi) = L$, where $L = a(1-\epsilon^2)$ is the "semi-latus rectum," a constant. A parabola corresponds to the $\epsilon \to 1$ limit, with $a(1-\epsilon^2) = L$ finite. You may find the material in §6.8.1 useful.