EQUILIBRIUM, FLOWS, AND ORBITS IN GENERAL RELATIVITY

NOTE: PROBLEM 2' IS OPTIONAL. PROBLEM 2 IS NOT.

1.) *Hydrostatic Equilibrium in GR.* Model a neutron star atmosphere with a simple equation of state: $P = K\rho^\gamma$, where P is pressure, ρ is mass density, γ is the adiabatic index and K is a constant. Assume that $g_{00} = -(1 - 2GM/rc^2)$, where M is the mass of the star and r is radius. If $\rho = \rho_0$ at the surface $r = r_0$, solve the equation of hydrostatic equilibrium to show that

$$\frac{1 + K\rho^{\gamma-1}/c^2}{1 + K\rho_0^{\gamma-1}/c^2} = \left(\frac{1 - R_S/r_0}{1 - R_S/r}\right)^\alpha$$

where $R_S = 2GM/c^2$ is the so-called Schwarzschild radius, and $2\alpha\gamma = \gamma - 1$. (Hint: See §4.6 of the notes.) What is the Newtonian limit of the above equation? Express your answer in terms of the speed of sound a, $a^2 = \gamma P/\rho$ and the potential $\Phi(r) = -GM/r$. (OPTIONAL: For those who have studied fluids, what quantity is being conserved in the Newtonian limit?)

2a.) *Practise with the Ricci Tensor.* Consider the 2D surface given by

$$z^2 = x^2 + y^2$$

where x, y, z are Cartesian coordinates in 3D Euclidian space. This represents a pair of cones centred on the origin, one cone opening upward, the other opening downward. The opening angle is 45° measured from the z axis. Justify this description.

2b.) A point in the 2D conic surface can be determined by R, the cylindrical radius of the point measured from the z-axis, and ϕ, the usual azimuthal angle. Show that the metric for the 2D surface in these coordinates is

$$ds^2 = 2dR^2 + R^2 d\phi^2$$

(Hint: Start with a standard metric in good old 3D Euclidian space, then enforce the constraint that $z^2 = x^2 + y^2 = R^2$. This is known as "embedding." The Nash Embedding Theorem states that pretty much any Riemannian hypersurface can always be embedded in some higher dimensional Euclidian space.)

2c.) Is this 2D surface curved, in the mathematical sense of having nonvanishing components of the curvature tensor $\mathcal{R}^\lambda_{\kappa\mu\nu}$? (We use $\mathcal{R}$ for the tensor, R for the radial coordinate.) Answer the question by showing that the metric of part 4b) can be transformed to new coordinates R', ϕ', for which

$$ds^2 = dR'^2 + R'^2 d\phi'^2$$

(The transformation law is *extremely* simple!) Why does this result alone answer the posed question? Can you give a physical interpretation of your mathematical transformation?

2d.) Next, consider a different 2D surface: $z = (\alpha/2)(x^2 + y^2)$ where α is an arbitrary constant parameter. Show that this is a paraboloid of revolution, i.e. a parabola spun around the z-axis. Prove that the metric within this surface is given by

$$ds^2 = (1 + \alpha^2 R^2)dR^2 + R^2 d\phi^2.$$

2e.) Prove that this surface *is* distorted by curvature. Calculate, for example, $\mathcal{R}_{\phi\phi}$ and show that it is not zero, but given by

$$\mathcal{R}_{\phi\phi} = -\frac{\alpha^2 R^2}{(1+\alpha^2 R^2)^2}$$

You should show en route that the only nonvanishing affine connection coefficients are

$$\Gamma^R_{RR} = \frac{\alpha^2 R}{1+\alpha^2 R^2}, \quad \Gamma^\phi_{\phi R} = \Gamma^\phi_{R\phi} = \frac{1}{R}, \quad \Gamma^R_{\phi\phi} = -\frac{R}{1+\alpha^2 R^2}$$

2'.) *Bondi Accretion: go with the flow. This is an* **optional** *(and long!) problem for students interested in relativistic fluids in astrophysics.* To get some practise working with the equations of GR, as well as some insight into relativistic dynamics, consider what is known as (relativistic) Bondi Accretion, the spherical flow of gas into a black hole. (The original Bondi accretion problem was Newtonian accretion onto an ordinary star.) We assume a Schwarzschild metric in the usual spherical coordinates:

$$g_{00} = -(1 - 2GM/rc^2),\ g_{rr} = (1 - 2GM/rc^2)^{-1},\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2 \sin^2\theta.$$

2'a.) First, let us assume that particles are neither created or destroyed. So particle number is conserved. If n is the particle number density in the local rest frame of the flow, then the particle flux is $J^\mu = nU^\mu$, where U^μ is the flow 4-velocity. Justify this statement, and using §4.5 in the notes, show that particle number conservation implies:

$$J^\mu_{\ ;\mu} = 0.$$

If nothing depends upon time, show that this integrates to

$$nU^r |g'|^{1/2} = \text{constant},$$

where g' is the determinant of $g_{\mu\nu}$ divided by $\sin^2\theta$, and U^r is...well, you tell me what U^r is.

2'b.) We move on to energy conservation, $T^{t\nu}_{\ ;\nu} = 0$. (Refer to §4.6 in the notes.) Show that the only nonvanishing affine connection that we need to use is

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}\frac{\partial \ln |g_{tt}|}{\partial r}$$

Derive and solve the energy equation. Show that its solution may be written

$$(P + \rho c^2) U^r U_t |g'|^{1/2} = \text{constant}$$

where $U_t = g_{t\mu} U^\mu$, and ρ is the total energy density of the fluid in its rest frame, including any thermal energy.

2'c.) We next define

$$\varpi = \mu n,$$

where μ is the rest mass per particle and ϖ is a Newtonian density. This is not to be confused with ρ, the true relativistic energy density divided by c^2. P and ϖ are assumed to be related by a simple power law relationship,

$$P = K\varpi^\gamma$$

where K is a constant, and γ is called the adiabatic index. This is not an entirely artificial problem: it is valid for cold classical particles ($\gamma = 5/3$) or hot relativistic particles ($\gamma = 4/3$). The first law of thermodynamics then tells us that the thermal energy per unit volume is

$$\epsilon = \frac{P}{\gamma - 1}$$

(You needn't derive that here, just use it!) Show that this implies:

$$\rho = \varpi + \frac{P}{c^2(\gamma - 1)}.$$

2'd.) Verify that

$$|g'| = r^4$$

and using $g_{\mu\nu}U^\mu U^\nu = -c^2$, show that

$$U_t = \left[c^2 - \frac{2GM}{r} + (U^r)^2\right]^{1/2}$$

(Take care to distinguish U^t and U_t.)

2'e.) With

$$a^2 = \gamma P/\varpi,$$

(this is the speed of sound in a nonrelativistic gas), combine our mass and energy conservation equations to show that

$$\boldsymbol{\left(c^2 + \frac{a^2}{\gamma - 1}\right)^2 \left(c^2 + U^2 - \frac{2GM}{r}\right) = \textbf{constant}.}$$

We have dropped the superscript r on U^r for greater clarity. How does a^2 depend upon ϖ? The other equation we shall use is just that of mass conservation itself. Show that this may be written as

$$4\pi\varpi r^2 U = \dot{m},$$

which defines the net, constant mass accretion rate $\dot{m} < 0$. With a^2 depending entirely on ϖ, and $\varpi = \dot{m}/(4\pi r^2 U)$, the equation in **boldface** becomes a single algebraic equation for U as a function of r, and the formal solution to our problem.

2'f.) Three final simple tasks for now:

i) Show that the constant on the right of the **bold** equation of problem (2e) is

$$c^2\left(c^2 + \frac{a_\infty^2}{\gamma - 1}\right)^2$$

where a_∞ is the sound speed at infinite distance from the black hole, if the gas starts accreting from rest.

ii) Show that the Newtonian limit of the equation is

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1}$$

where v is the ordinary velocity, not the 4-velocity. This is a statement that a quantity known as enthapy (energy plus the work done by pressure) is conserved. This is the original nonrelativistic Bondi 1952 solution for accretion onto a star.

iii) Show that as r approaches the Schwarzschild radius $R_S = 2GM/c^2$, then if $a \ll c$ everywhere, then dr/dt satisfies the condition of a "null geodesic," a fancy way to say the inflow follows the equation of light:

$$\frac{dr}{dt} = -c(1 - R_S/r).$$

Like stalled photons, from the point of view of a distant observer, the flow never crosses R_S.

3a.) *Kinematic and gravitational redshifts.* One of the most important observational black hole diagnostics is a calculation of the radiation spectrum from the surrounding disc. In particular we are interested in how the frequency of a photon is shifted due to space-time distortions and relativistic kinematics. Show that:

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu(R)V^\mu(R)}{p_\mu(E)V^\mu(E)}$$

where R denotes the received the photon and E the emitted photon, ν is a frequency (not an index here!), p_μ a covariant photon 4-momentum, and V^μ is the normalised (to c) 4-velocity in the form $(dt/d\tau, d\mathbf{x}/cd\tau)$ for the emitted material (E) or the distant observer at rest (R).

3b.) In the problem at hand, the observer views the disc edge-on, in the plane of the disc. The gas moves in circular orbits. Here is the disc spherical midplane ($\theta = \pi/2$), viewed from above:

$\odot$ $\longrightarrow$ observer $\gtrdot$

Show that in t, r, θ, ϕ coordinates for the $0, 1, 2, 3$ components,

$$V^\mu(R) = (1, 0, 0, 0), \quad V^\mu(E) = V_E^0(1, 0, 0, d\phi/cdt), \text{ with } V_E^0 = dt/d\tau$$

Then, using $g_{\mu\rho}V^\mu V^\rho = -1$, conclude that

$$V_E^0 = (1 - 3GM/rc^2)^{-1/2}$$

You may use the result from problem (5c) below that $\Omega^2 = (d\phi/dt)^2 = GM/r^3$. (You will prove it later!)

3c.) Finally, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} \left(1 + \frac{\Omega p_\phi(E)}{cp_0(E)}\right)^{-1}, \quad \Omega^2 = GM/r^3.$$

A result from problem (4) of Problem Set 1 may be useful.

From disk material moving at right angles across the line of sight, ν_R/ν_E reduces to

$$(1 - 3GM/rc^2)^{1/2}.$$

Why? From disk material moving precisely along the line of sight, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - 3GM/rc^2\right)^{1/2} / \left(1 \pm (rc^2/GM - 2)^{-1/2}\right)$$

(Hint: $g^{\nu\rho}p_\nu p_\rho = 0$. You may ignore deflection of the photon trajectories and assume they move in straight lines.) Interpret the $\pm$ sign. In general, the photon paths must be calculated from the dynamical equations to determine the $p(E)$ ratio.

4a.) *The perihelion advance of Mercury.* In the notes we found that the differential equation for $u = 1/r$ for Mercury's orbit could be written as follows. $u = u_N + \delta u$ with the Newtonian solution u_N given by

$$u_N = (GM/J^2)(1 + \epsilon \cos\phi)$$

and the differential equation for δu is

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3(GM)^3}{c^2 J^4}(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi).$$

Show that this is equivalent to solving the real part of the equation

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a(b + 2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2)$$

where $a = 3(GM)^3/(c^2J^4)$ and $b = 1 + \epsilon^2/2$.

To solve this, try a solution of the form

$$\delta u = A_0 + A_1\phi e^{i\phi} + A_2 e^{2i\phi}$$

where the A's are constants. Why do we need an additional factor of ϕ in the A_1 term?

4b.) Show that the solution for $u = u_N + \delta u$ is

$$u = \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon\cos\phi + \epsilon a\phi\sin\phi$$

Since a is very small, show that this equivalent to

$$u = ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}[1 + \epsilon(\cos\phi(1 - \alpha))]$$

where

$$\alpha = aJ^2/GM = 3(GM/Jc)^2$$

4c.) In the equation for u, the first two terms in a cause tiny (and unmeasurable) distortions in the shape of the ellipse, but do not affect the 2π perodicity in ϕ of the orbit. Show however that the final term, proportional to GM/J^2, results in a periastron advance of

$$\Delta\phi = 6\pi \left(\frac{GM}{cJ}\right)^2$$

each orbit. This is the classic Einstein result.

5a.) *Black hole orbits.* In Newtonian theory, the energy equation for a test particle in orbit around a point mass is

$$\frac{v^2}{2} + \frac{l^2}{2r^2} - \frac{GM}{r} = \mathcal{E}$$

where r is radius, v is the radial velocity, l the angular momentum per unit mass, $\mathcal{E}$ the constant energy per unit mass, and $-GM/r$ is of course the potential energy. For the Schwarzschild solution show that the integrated geodesic equation may also be written in the form

$$\frac{v_S^2}{2} + \frac{l_S^2}{2r^2} + \Phi_S(r) = \mathcal{E}_S$$

where r is the standard radial coordinate, l_S and $\mathcal{E}_S$ are constants, $\Phi_S(r)$ a modified potential function, and $v_S = dr/d\tau$. Determine l_S, Φ_S and $\mathcal{E}_S$ in terms of the fundamental angular momentum and energy constants J and γ_∞ from lecture (or the notes) and GM or c as needed. The form of l_S, $\mathcal{E}_S$, and Φ_S should be chosen to reveal how they go over to their Newtonian counterparts in the appropriate limit.

5b.) Sketch the effective potential $l_S^2/2r^2 + \Phi_S(r)$. Prove that there is always a potential minimum in Newtonian theory, but that this is not the case in general relativity. What is the mathematical condition for the existence of a potential minimum for the effective potential, and what does it mean physically if it does not exist?

5c.) Show that for the Schwarzschild metric, circular orbits satisfy

$$\Omega^2 = \frac{GM}{r^3},$$

exactly the Newtonian form. Here $\Omega(r) \equiv d\phi/dt$, where $dt \neq d\tau$ is the proper time interval at *infinity.* Derive expressions for J^2 and γ_∞^2 in terms of GM, c^2 and r.

5d.) Below what value of r does the effective potential not have any local extrema? (Answer: $6GM/c^2$.)