Holography, Finite-Temperature QFT and Hydrodynamics

MITP Summer School

Mainz, July 2019

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- Plan
  - introduction and motivation
  - thermodynamics and transport
  - two essential tools
  - relativistic hydro
  - Kubo formulae

<u>Motivation</u>: want to understand physical properties of macro systems based on their microscopic description („from first principles")

N interacting constituents $\xrightarrow[\text{limit}]{\text{TD}}$ <u>Equil. state</u>

Dynamics governed by H (or $\mathcal{L}$) $\longrightarrow$ <u>Few conserved quantities</u> (E_{TOT}, L_{TOT}, Q_{TOT})

Ergodicity etc – not obvious (see e.g. H. S. Dumas ~~book~~ book about KAM theorem)

Practical needs: e.g. water – submarines or atmosphere of Jupiter or exoplanets (communications).

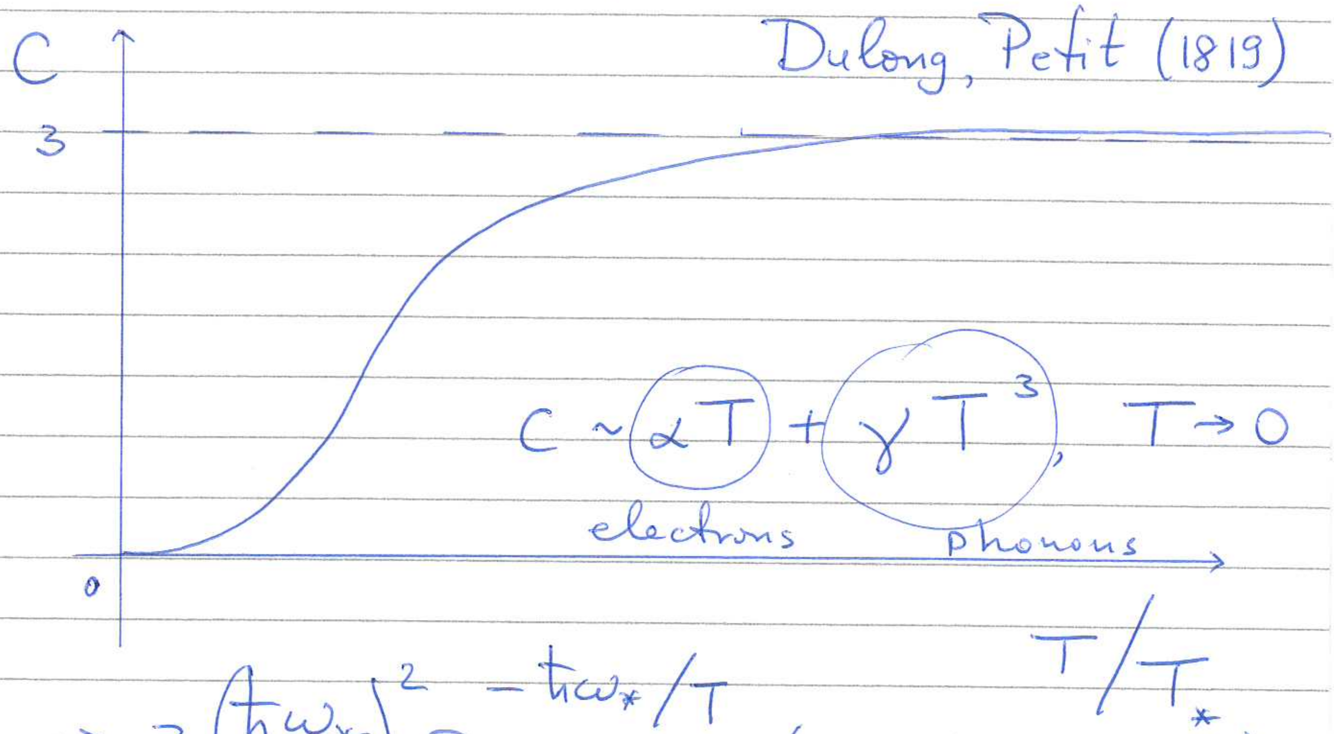
Need:

- thermodynamic properties
(eos, phase trans., specific heat etc)

- transport properties
(viscosity, conductivities, diffusion consts etc)

- strongly non-equil. behaviour

Example: specific heat of solids

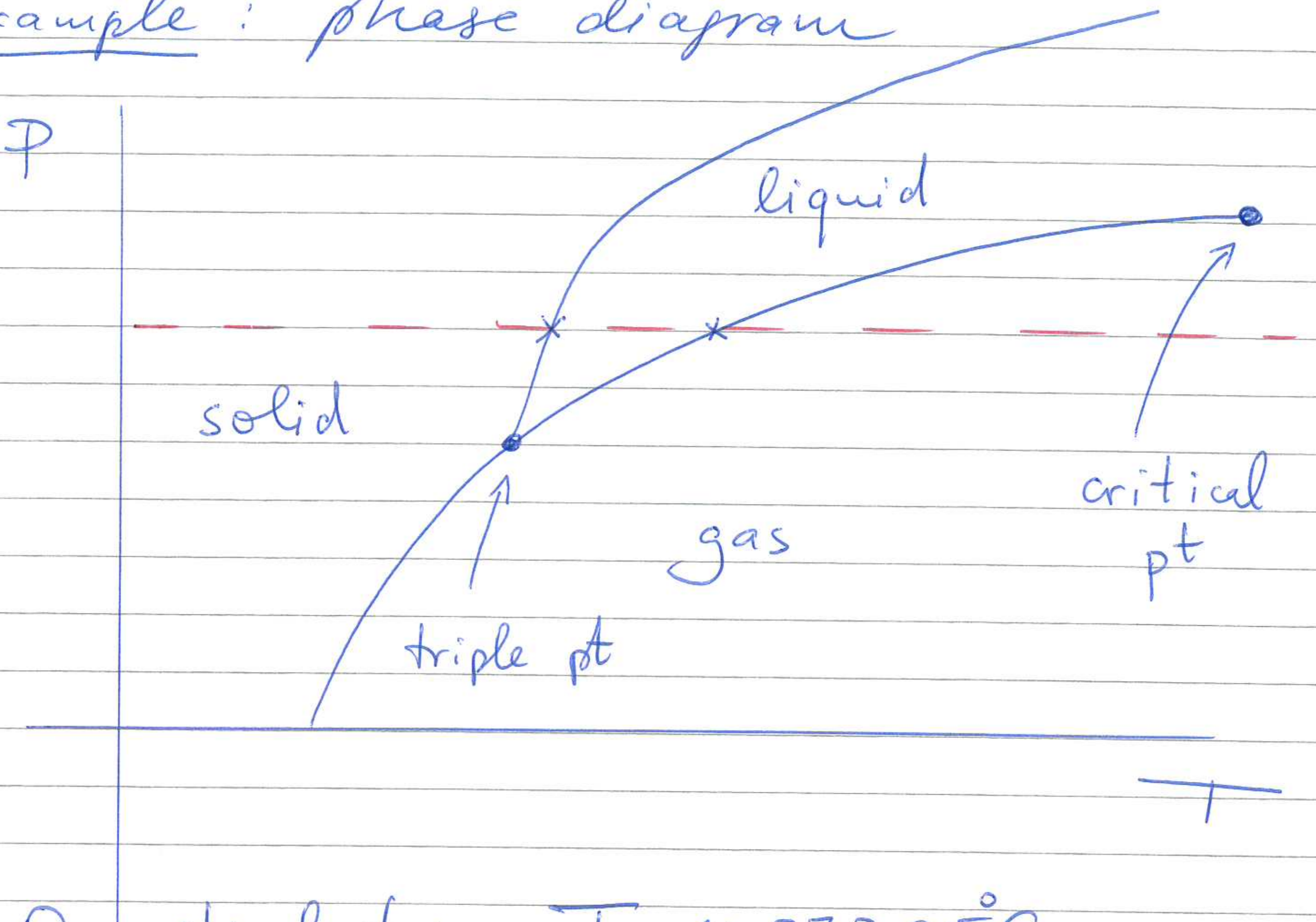


- $C \to 3\left(\frac{\hbar\omega_*}{T}\right)^2 e^{-\hbar\omega_*/T}$ (Einstein, 1907) $\omega(k) = \omega_*$

- $C \to \frac{12}{5}\frac{\pi^4}{\Theta_*^3} T^3$ (Debye, 1912) $\omega(k) = c_s k$

- Born, von Kármán (1912-1913) $\omega(k) = \ldots$

Example : phase diagram



H_2O critical pt : $T_c \approx 373.95°C$

$P_c \approx 221$ atm

Can this be predicted from "first principles"?

Example : linear resistivity of "strange metals" at low T

- Landau Fermi-liquid th : $\rho \sim T^2$ (almost all metals)

- "Strange metals" : $\rho \sim T$ (?)

Standard tools

- $\hat{\rho}(t)$: density matrix operator

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]$$

$$\langle \hat{O} \rangle = \mathrm{tr}\, \hat{\rho} \hat{O} \qquad (\mathrm{tr}\, \hat{\rho} = 1)$$

- TD equil: $\hat{\rho} = e^{-\beta \hat{H} + \mu_\alpha \hat{Q}^\alpha} / Z$

$$Z = \mathrm{tr}\, e^{-\beta \hat{H} + \mu_\alpha \hat{Q}^\alpha}$$

$\beta = 1/T$ $\quad \hat{Q}^\alpha$: conserved charges

e.g. $\hat{Q} = \hat{N}$ (number of particles)
for non-rel. system or $\hat{Q} = \hat{Q}_B$
in QCD.

<u>Remark</u>: GGE in integrable models (Rigol) and generalised hydro (Doyon).

- All TD: $\Omega = -T \ln Z$

$$\mathcal{E} = \langle \hat{H} \rangle = -\frac{\partial \Omega}{\partial \beta} \; ; \; Q_\alpha = \langle \hat{Q}_\alpha \rangle = -\frac{\partial \Omega}{\partial \mu_\alpha}$$

$$S = -\partial\Omega/\partial T\big|_{V,\mu} \quad \text{(entropy)}$$

$$P = -\frac{1}{V}\Omega \quad \text{(pressure) etc.}$$

This works very well in textbook examples

<u>Example</u> : N diatomic molecules in $d_s = 3$ dim (H_2 or N_2 etc)

$$\hat{\mathcal{H}} = \sum_{i=1}^{N}\left(\frac{\hat{P}_i^2}{2m} + \hat{\mathcal{H}}_{internal,i}\right) + \cancel{\sum_{i<j}\hat{\mathcal{O}}(i,j)}$$

ideal gas approx

$$\hat{\mathcal{H}}_{internal} = \hat{\mathcal{H}}_{rot} + \hat{\mathcal{H}}_{osc} + \cancel{\hat{\mathcal{H}}_{electr}} + \dots$$

$$Z = Z_0 \, Z^N_{internal}$$

monoat. ideal gas

$$Z_{intern} = Z_{rot}\, Z_{osc} \cdot Z_{el} \cdots$$

$$Z_{osc} = \mathrm{tr}\,\hat{\rho} = \sum_n \langle n | e^{-\beta\hat{\mathcal{H}}_{osc}} | n\rangle$$

$$\hat{\mathcal{H}}_{osc}|n\rangle = E_n|n\rangle \qquad E_n = \hbar\omega_0\,(n + 1/2)$$

$$n = 0, 1, \dots$$

$$Z_{osc} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega_0 n} e^{-\beta\hbar\omega_0/2} = \frac{1}{2\sinh\frac{\beta\hbar\omega_0}{2}}$$

Z_{rot} : $$E_{lm} = \frac{\hbar^2}{2I} l(l+1)$$

$$l = 0, 1, 2 \ldots \quad m = \overline{-l, l}$$

$$Z_{rot} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-\frac{\hbar^2\beta}{2I} l(l+1)} =$$

$$= \sum_{l=0}^{\infty} (2l+1) e^{-\frac{\hbar^2\beta}{2I} l(l+1)}$$

<u>Exercise</u> : show that in $d_s = 2$

$$Z_{rot} = \sum_{l \in \mathbb{Z}} e^{-\pi\bar{\beta} l^2} = \Theta_3(\beta, 0) \quad (\text{Jacobi})$$

$$\bar{\beta} = \frac{\hbar^2 \beta}{2\pi I}.$$

<u>Example</u> : relativistic ideal gas

$$\varepsilon_p = \sqrt{\bar{p}^2 + m^2}$$

$$\Omega = \pm T \sum_{\bar{p}} \ln\left(1 \mp e^{\frac{\mu - \varepsilon_p}{T}}\right)$$

Exercise: show that

$$P = g_s \frac{m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \varkappa^{n-1} \frac{K_2\left(\frac{m}{T} n\right)}{n^2} e^{n\mu/T}$$

mod Bessel f.

$\varkappa = +1$ Bosons $\qquad \varkappa = -1$ Fermions

$g_s = 2s+1 \qquad (g_s = 2$ for photon, gluon$)$

Classical Boltzmann gas: $e^{|\mu|/T} \ll 1$

$$\Rightarrow P \longrightarrow g_s \frac{m^2 T^2}{2\pi^2} K_2\left(\frac{m}{T}\right) e^{\mu/T}$$

$$\left(K_2(z) \underset{z \to 0}{\to} \frac{2}{z^2},\right)$$

Non-rel. limit:

$$P = T g_s \left(\frac{mT}{2\pi}\right)^{3/2} e^{\mu_{NR}/T}$$

$\mu_{NR} \equiv \mu - m \qquad$ (Note: $c = 1$, $\hbar = 1$).

Exercise: consider QCD

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu}_{a} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

and show that for ideal relativistic quantum gas of N_c^2-1 gluons and N_f fund. fermions the pressure is

$$P_{SB} = \frac{(N_c^2-1)\pi^2}{45}\left(1 + \frac{7}{4}\frac{N_f N_c}{N_c^2-1}\right)T^4 + O(m^2T^2)$$

$\uparrow$

Stefan-Boltzmann

- In QFT, Z can be computed via path integral.

$$Z = \int d\phi \, \langle \phi | e^{-\beta \hat{H}} | \phi \rangle$$

We know how to represent the amplitude

$$\langle \phi_f ; t_f | e^{-i\hat{H}(t_f - t_i)} | \phi_i ; t_i \rangle$$

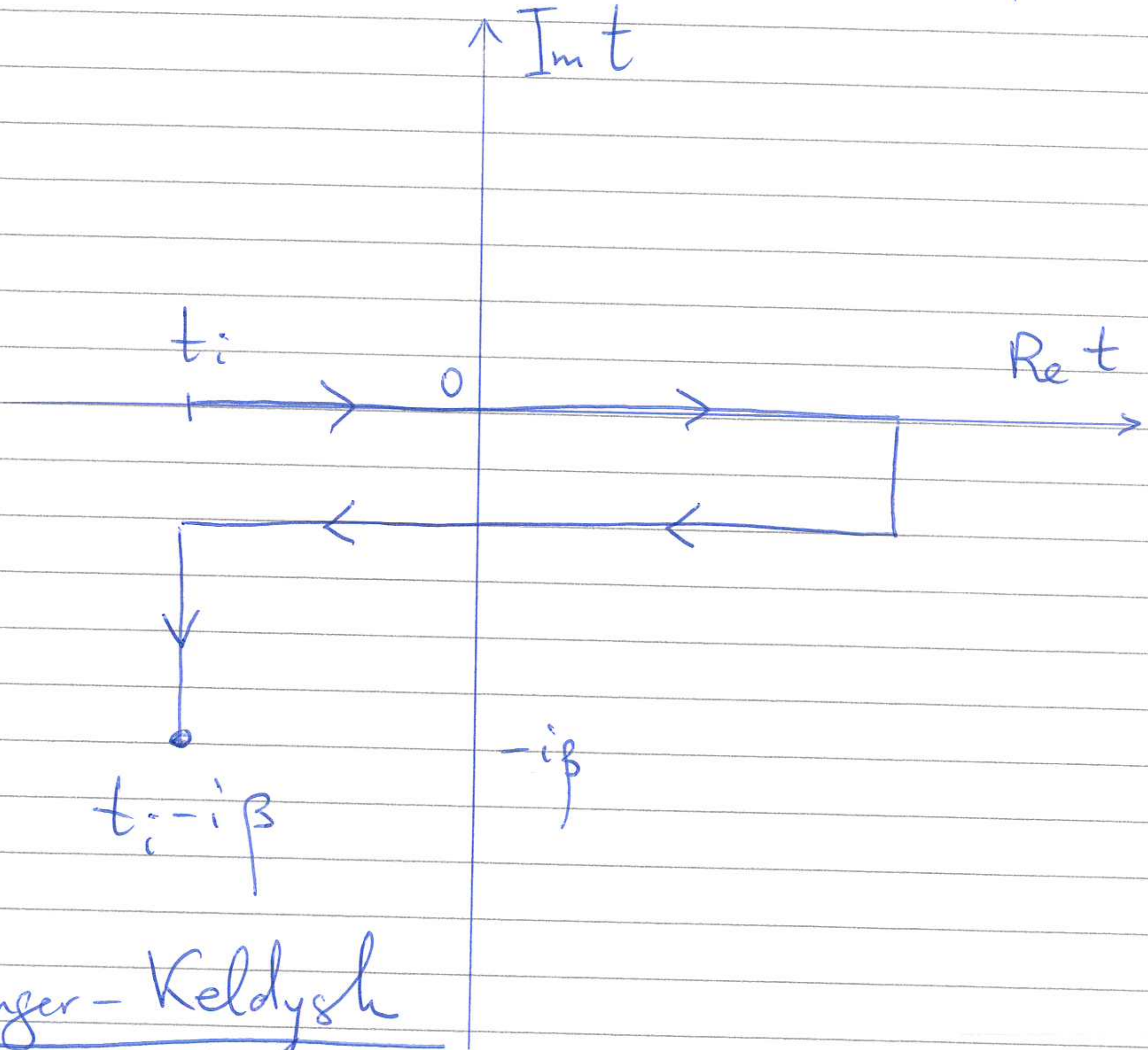
via path integral (Feynman)

Identify: $t_f - t_i = -i\beta$

$$\phi(t_f) = \phi(t_i) \quad (*)$$

$$Z[J] = \int [d\phi]\, e^{i\int_C d^4x\,(\mathcal{L} + J\phi)}$$

Here C: contour from t_i to $t_i - i\beta$:



Schwinger–Keldysh

- Imaginary t contour: $t = -i\tau$

  Matsubara: „im time formalism"
- Real-t contour

- conditions on fields

$$\phi(t_i, \bar{x}) = \eta \, e^{\beta \mu} \phi(t_i - i\beta, \bar{x})$$

$\eta = \pm 1$ (Bosons / Fermions)
(Index theorems)

- KMS conditions for correlators

This can be used to compute TD quantities e.g. of QCD (pressure etc) either via pert. theory (in $\alpha_s = g^2/4\pi$) or using lattice (in Euclid).

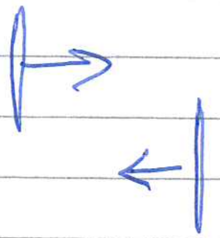
$$P_{QCD} = \frac{\pi^2 T^4}{45}(N_c^2-1)\left[1 + \frac{7}{4}\frac{N_f N_c}{N_c^2-1} - \frac{5}{4}\left(N_c + \frac{5}{4}N_f\right)\frac{\alpha_s}{\pi}\right.$$

$$\left. + 30\left(\frac{N_c}{3} + \frac{N_f}{6}\right)^{3/2}\left(\frac{\alpha_s}{\pi}\right)^{3/2} + O\left(\alpha_s^2, \alpha_s^2 \ln \alpha_s, \alpha_s^{5/2}, \alpha_s^3 \ln \alpha_s \ldots\right)\right].$$

- non-analytic terms present
- resummation of ring diagrams etc needed

Nuclear matter, quark-gluon plasma

Heavy ion collisions (RHIC, LHC, FAIR, NICA)



$^{197}_{79}Au$ or $^{207}_{82}Pb$

- crude estimate of energy density

— energy of the beam (RHIC) $\sim 200\ GeV/_{\text{nucleon pair}}$

— $R_{Au} \sim 7\ fm$ $(1\ fm = 10^{-13}\ cm)$

— $\mathcal{E}_{TOT} \sim 200 \frac{GeV}{nucl}\ 200\ nucl \sim 4 \cdot 10^4\ GeV$

— $\epsilon = \mathcal{E}_{TOT}/V_3 \sim \dfrac{4 \cdot 10^4\ GeV}{\frac{4}{3}\pi R^3\ fm^3} \sim 30\ GeV/_{fm^3}$

— More accurate estimates $\rightarrow \epsilon \sim 5\ GeV/_{fm^3}$

What is known about this regime theoretically (pert. theory + LQCD)?

Thermodynamics and transport in QCD from first principles

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

Param: $N_c = 3$ $SU(N_c)$

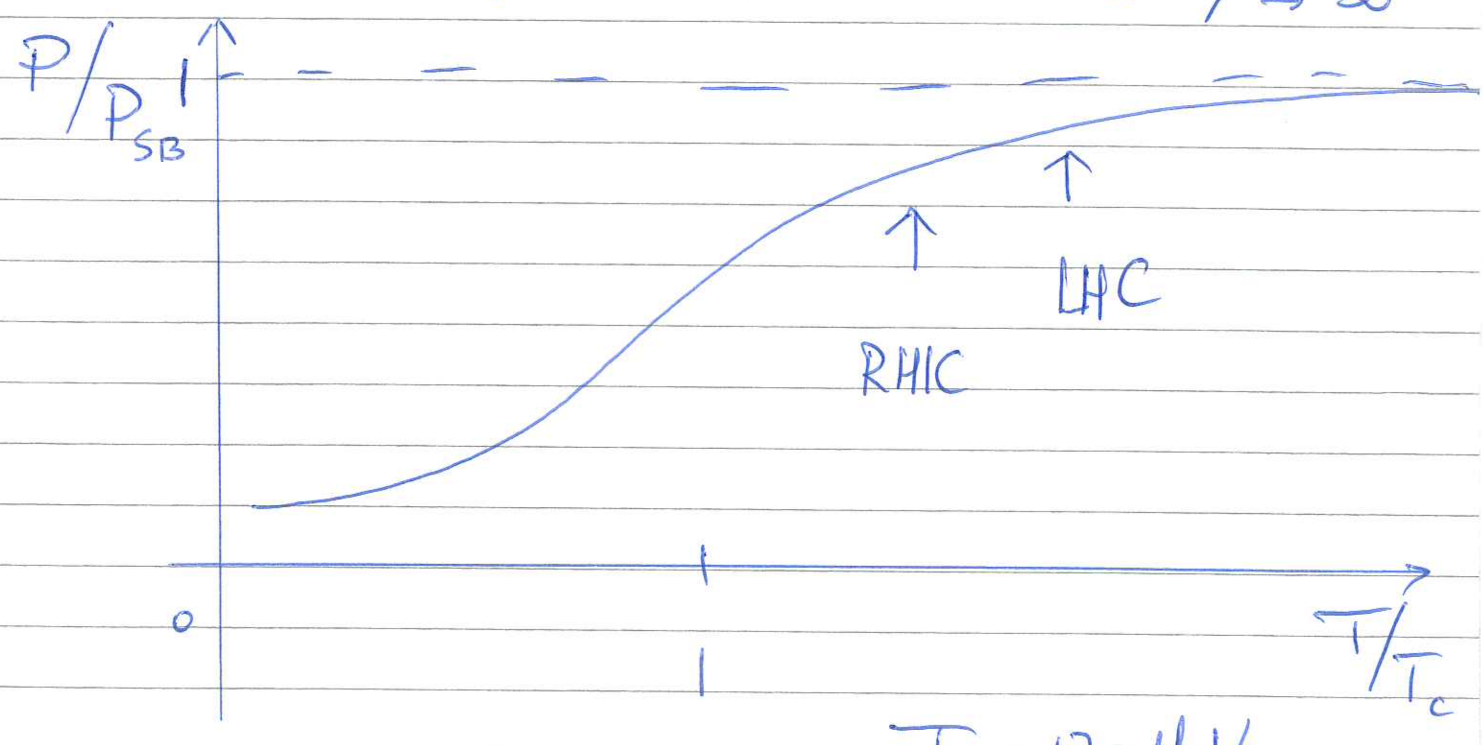
$N_f = 6$

$\alpha_s = g^2_{YM}/4\pi$ is not a fixed param:

$\alpha_s \to 0$ at $E \to \infty$ (asympt. freedom),

i.e. $E/\Lambda_{QCD} \to \infty$

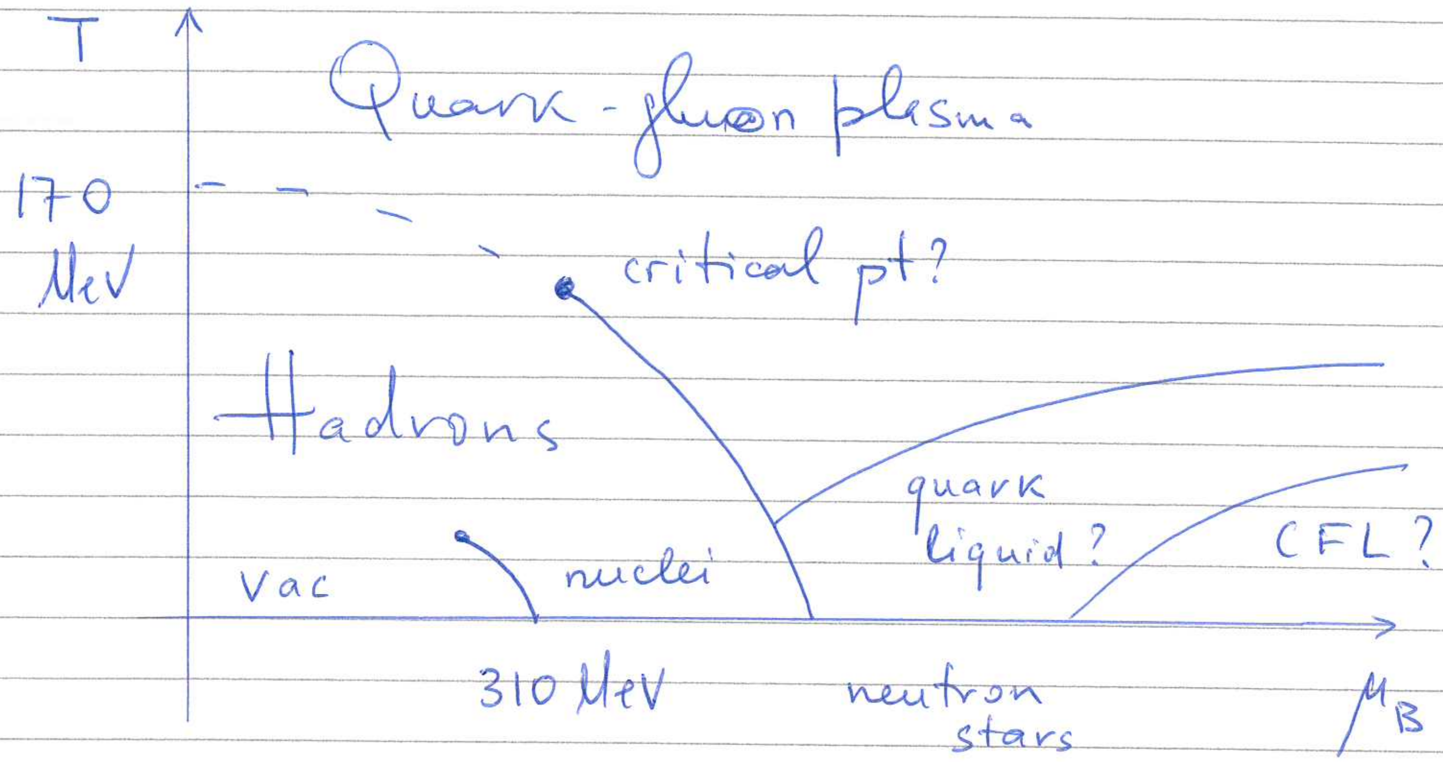
$\Rightarrow$ expect ideal gas of quarks and gluons at $T \to \infty$



$T_c \sim 170$ MeV

($\epsilon_c \sim 1$ GeV: ϵ/ϵ_{SB} is similar).

QCD phase diagram



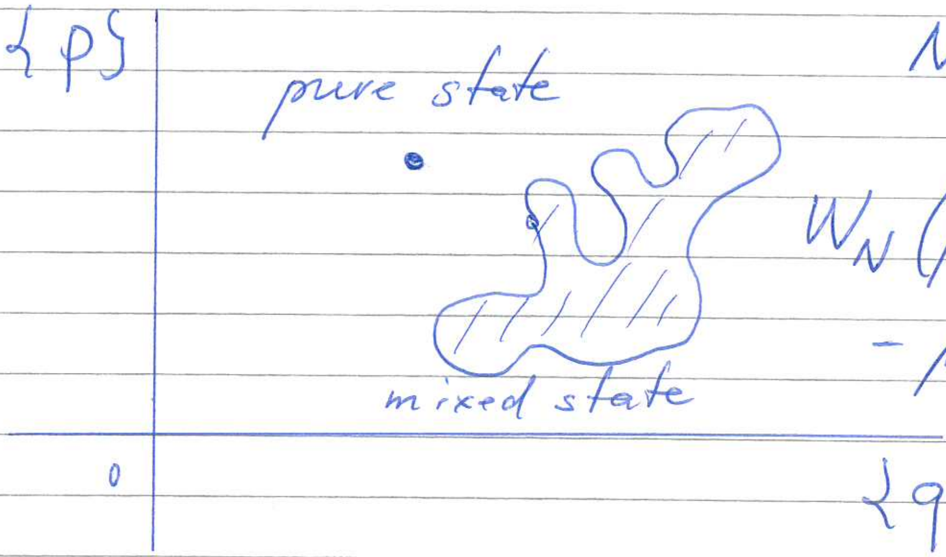
Evolution of QGP at scales of $(1-10)\cdot 10^{-24}$ sec is described by rel. Navier-Stokes eqs. Need ways to compute transport (e.g. visc.) of QCD. How?

- LQCD + analytic continuation (difficult)
- Pert. theory (real-time) - difficult at $\alpha_s \sim 1$
- Kinetic theory (not applicable)

Kinetic theory

- Classical / quantum
- Classical: BBGKY → Boltzmann

  Vlasov eq.



N particles

$W_N(p_1 \cdots p_N; q_1 \cdots q_N; t)$

– prob. distribution

$$\frac{\partial W_N}{\partial t} = \{H, W\}_{PB}$$

since $\dfrac{dW_N}{dt} = 0$ (Liouville th 1838)

$$F_1(t, p_1, q_1) = V \int W_N \, dp_2 \cdots dp_N \, dq_2 \cdots dq_N$$

$$F_2(t, p_1, p_2, q_1, q_2) = V^2 \int W_N \, dp_3 \cdots dp_N \, dq_3 \cdots dq_N$$

. . . . . .

$$F_s(t, p_1 \cdots p_s, q_1 \cdots q_s) = V^s \int W_N \, dp_{s+1} \cdots dp_N \, dq_{s+1} \cdots dq_N$$

For:

$$H = \sum_i \left(\frac{p_i^2}{2m} + U_{ext}(q_i) \right) + \sum_{i<j} \Phi(|q_i - q_j|)$$

$$\frac{\partial F_1}{\partial t} + \frac{p_i}{m}\frac{\partial F_1}{\partial q_i} - \frac{\partial U_{ext}}{\partial q_i}\frac{\partial F_1}{\partial p_i} = \frac{N}{V}\int \frac{\partial \Phi(|q-q'|)}{\partial q_i} \cdot \frac{\partial F_2(t,q,q',p,p')}{\partial p_i} dq' dp'$$

BBGKY (e.g. Bogolyubov, 1946)

$$\frac{\partial F_s}{\partial t} + \sum_{k=1}^{s}\left(\frac{p_i^{(k)}}{m}\frac{\partial F_s}{\partial q_i^{(k)}} - \frac{\partial U_{ext}}{\partial q_i^{(k)}}\frac{\partial F_s}{\partial p_i^{(k)}} \right) =$$

$$= \frac{1}{v}\sum_{k=1}^{s} \int \frac{\partial \Phi(|q_i^{(k)} - q_i^{(s+1)}|)}{\partial q_i^{(k)}} \frac{\partial F_{s+1}}{\partial p_i^{(k)}} dq_{s+1} dp_{s+1}$$

$v \equiv V/N$.

Closed eq for F_1 - ?

Let Φ_0 - strength of Φ, r_0 - effective radius of interactions, τ - timescale of kin eq., $l = l_{mfp}$, $mu_0^2 = 3kT$, $u_0 = \frac{l}{\tau}$ defines T scale.

Decoupling of BBGKY

- $r_0^3/v \ll 1$ (dilute system)

- $\phi_0 \ll kT$ (with $r_0^3 \sim v$) : weakly coupled system

- $\Phi_0 \ll kT$ (with $\frac{r_0^3}{v} \sim \frac{kT}{\Phi_0} \gg 1$)

  Debye - Hückel

- $\phi_0 \ll kT$, $r_0^3 \ll v$ weak inter., dilute system

$\boxed{\text{In all cases,} \quad \tau \gg \frac{\hbar}{kT}.}$ Bogolyubov Gurov 1947 Gurov, 1966

(E.g. air at room $T \sim 300K$: $\frac{\hbar}{kT} \sim 10^{-13}$ sec

$\tau \sim 10^{-10}$ sec $-$ OK).

$$F_2(t, p_1, p_2, q_1, q_2) \xrightarrow[|q_2 - q_1| \to \infty]{} F_1(t, p_1, q_1) F_1(t, p_2 q_2)$$

$\Rightarrow$ Boltzmann

Schemes of decoupling: Kirkwood, Bogolyubov, others.

Linear response theory

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F(t) \qquad (\gamma > 0)$$

$$x(t) = x_0(t) + \int_{-\infty}^{\infty} G(t-t') F(t')\, dt',$$

where $\quad L G = \delta(t-t')$

$$L = \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \qquad \left(L \text{ is transl. invar in } t \right)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} X(\omega)\, d\omega$$

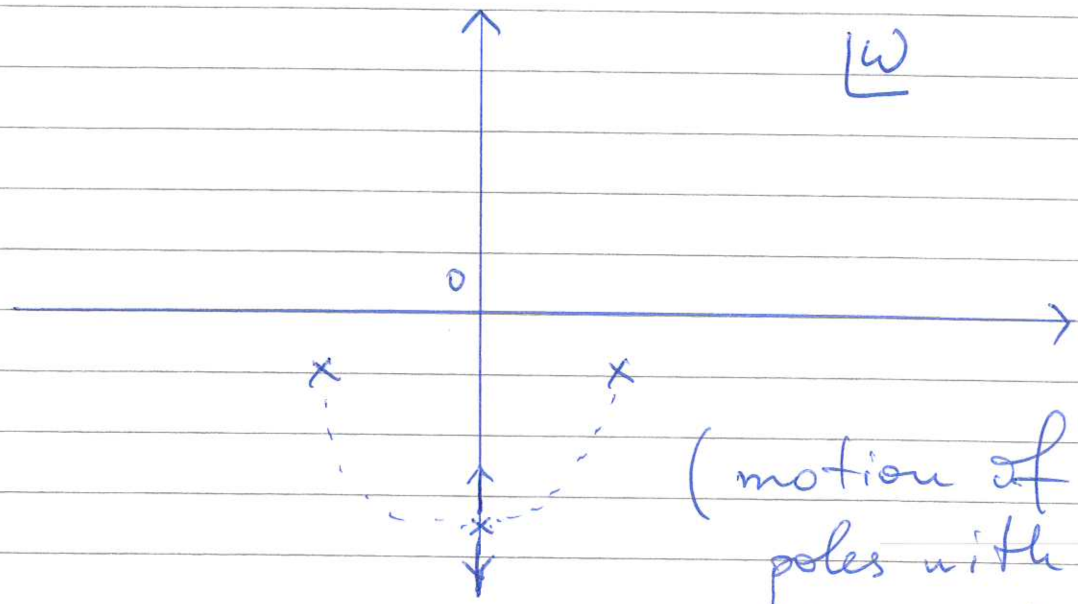
$$\delta(t-t') = \frac{1}{2\pi} \int e^{-i\omega(t-t')}\, d\omega$$

$$\Rightarrow \quad G(\omega) = -\frac{1}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$\omega_\pm = \pm\sqrt{\omega_0^2 - \gamma^2} - i\gamma$$

$$X(\omega) = X_0(\omega) + G(\omega)F(\omega)$$

or $$\boxed{\delta X(\omega) = G(\omega)F(\omega)}$$



(motion of poles with γ changing)

$G(\omega)$ analytic in $\mathrm{Im}\,\omega > 0$ plane

$\Rightarrow G(t-t') = 0$ for $t-t' < 0$:

$$\delta x(t) = \int_{-\infty}^{\infty} G(t-t')f(t')\,dt' = \int_{-\infty}^{t} G(t-t')f(t')\,dt'$$

(causality)

$$G(t-t') \equiv G_R(t-t') = \theta(t-t')\,e^{-\gamma(t-t')}\frac{\sin\left[\sqrt{\omega_0^2-\gamma^2}\,(t-t')\right]}{\sqrt{\omega_0^2-\gamma^2}}$$

Retarded Green's function

In QM or QFT:

$$\hat{H}_0 \to \hat{H}_0 + \delta\hat{H}(t)$$

$$\delta\hat{H} = -\int d^3x \, \lambda_a(t,\bar{x})\,\hat{\mathcal{O}}_a(t,\bar{x})$$

Then (time-dep. pert. theory to linear order in λ):

$$\delta\langle\hat{\mathcal{O}}_a(t,\bar{x})\rangle = -\int dt'\,d^3x'\,G^R_{ab}(t-t',\bar{x}-\bar{x}')\,\lambda_b(t',\bar{x}')$$

where $\delta\langle\hat{\mathcal{O}}\rangle = \langle\hat{\mathcal{O}}\rangle_{H_0+\delta H} - \langle\hat{\mathcal{O}}\rangle_{H_0}$

$$G^R_{ab}(t-t',\bar{x}-\bar{x}') = -i\theta(t-t')\left\langle\left[\hat{\mathcal{O}}_a(t,\bar{x}),\hat{\mathcal{O}}_b(t',\bar{x}')\right]\right\rangle_{H_0}$$

In Fourier space,

$$\delta\langle\hat{\mathcal{O}}_a(\omega,\bar{q})\rangle = -G^R_{ab}(\omega,\bar{q})\,\lambda_b(\omega,\bar{q})$$

E.g. $\hat{\mathcal{O}} = \hat{J}^\mu$ or $\hat{T}^{\mu\nu}$ (conserved currents)

then $\lambda = A_\mu$ or $h_{\mu\nu}$

See: Le Bellac; Kovtun 1205.5040
Kiritsis, Ch. 15 (2nd ed).

Transport in QFT from „first principles"

Effective theory at large dist. and large times ≡ fluid dynamics

Equil: $E_{TOT} = \text{const}$ $Q_{TOT} = \text{const}$

Near-eq: $\epsilon(t,\bar{x}) = E/V$, $q(t,\bar{x}) = Q/V$

– densities of conserved charges

More precisely: $\langle \hat{T}^{\mu\nu} \rangle \equiv T^{\mu\nu}(t,\bar{x})$

exp. value in a state
described by $\hat{\rho}(t,\bar{x})$

In equil: $\langle \hat{T}^{\mu\nu} \rangle_{T,\mu} = \mathrm{tr}\, \hat{\rho}_{eq} \hat{T}^{\mu\nu} =$

$$= T^{\mu\nu}_{eq} = \begin{pmatrix} \epsilon & & & 0 \\ & P & & \\ & & P & \\ 0 & & & P \end{pmatrix} \quad \text{for} \qquad (1)$$

isotropic homogeneous system in rest frame.

Covariant expression: in a frame with const $u^\mu = (\gamma c, \gamma \vec{v})$:

$$T^{\mu\nu}_{eq} = T^{\mu\nu}_{\parallel} + T^{\mu\nu}_{\perp} =$$

$$= \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} \qquad (2)$$

$$\Delta^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1).$$

$$u_\mu \Delta^{\mu\nu} = 0$$

<u>Exercise:</u> derive (2) using (1). Show that

$$\Delta_{\mu\nu} \Delta^\nu{}_\sigma = \Delta_{\mu\sigma}.$$

We now assume $u^\mu = u^\mu(x)$. Then $T^{\mu\nu}$ will contain derivatives of $u^\mu(x)$. Assume gradients are small in <u>some</u> sense $\Rightarrow$ $\partial_\mu u_\nu$ („slow variables")

$\Rightarrow$ gradient expansion

- assumes analyticity (no $\omega^{3/2}$ terms in Fourier)

- convergence - ?

$$T^{\mu\nu} = T^{\mu\nu}_0 + T^{\mu\nu}_1 + T^{\mu\nu}_2 + O(\partial^3)$$

number of deriv. involved

$$T_0^{\mu\nu} = \epsilon u^\mu u^\nu + p \Delta^{\mu\nu}$$

$$T_1^{\mu\nu} = -\eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right)$$
$$- \zeta \Delta^{\mu\nu}\partial_\lambda u^\lambda$$

- In curved space-time: $\eta_{\alpha\beta} \to g_{\alpha\beta}$
  $\partial_\alpha \to \nabla_\alpha$

Combining $\partial_\mu T^{\mu\nu} = 0$ and $T^{\mu\nu} = \sum_{k=0}^{\infty} T_k^{\mu\nu}$,

we get eom:

$k = 0$ Euler (relativ.)

$k = 1$ Navier-Stokes (rel)

$k = 2$ Burnett (rel)

etc

- $k = 3$ case is known partially in rel. hydro (Grozdanov, Kaplis)
  $k > 3$ unknown except for special cases such as low-dim Bjorken flow.
- Number of transport coeffs with k?

Similar eqs can be written in case of a conserved current j^μ:

$$\begin{cases} \partial_\mu j^\mu = 0 \\ j^\mu = j_0^\mu + j_1^\mu + \ldots = \rho u^\mu - D\Delta^{\mu\nu}\partial_\nu \rho + \ldots \end{cases}$$

convection diffusion

In the rest frame of fluid: $k = x, y, z$

$$j^k = -D\,\partial_k \rho$$ Fick's Law of diffusion

Hydro: d.o.f. - densities of conserved charges (e.g. ρ)

eom: $\partial_\mu j^\mu = 0 \oplus j^\mu = \sum_{k=0} j^\mu_{(k)}$

$$\Rightarrow \partial_t \rho + D\nabla^2 \rho = 0 \quad \text{(diffusion eq)}$$

We now use linear response formula with $\hat{O} = \hat{T}_{xy}$ and $\vec{q}$ along z in isotropic system

$$\delta T_{xy}(\omega, \bar{q}) = - G^R_{T_{xy} T_{xy}}(\omega, \bar{q})\, h_{xy}(\omega, \bar{q}),$$

where h_{xy} is the metric perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(t)$$

with $h_{xy} \neq 0$ only (see 0704.0240)

But for small ω, $|\bar{q}|$, we have an explicit expression for T_{xy} as deriv. expansion (= series in powers of $\omega, |\bar{q}|$).

Since $h_{\mu\nu}(t)$ does not dep. on $\bar{x}$, rot. symm. is preserved $\Rightarrow u^i = 0 \quad i = x, y, z$.

$\Rightarrow u^\mu = (1, \vec{0})$. But

$$\nabla_x u_y = \partial_x u_y - \Gamma^\lambda_{xy} u_\lambda = -\Gamma^t_{xy} u_t = \Gamma^t_{xy}.$$

$$\nabla_y u_x = \Gamma^t_{yx} = \Gamma^t_{xy}. \quad \left(\Gamma^t_{xy} = \tfrac{1}{2} \partial_t h_{xy} \right)$$

So for $\delta T_{xy} = T_{xy} - \underset{\simeq 0}{T^{eq}_{xy}}$

$$\delta T_{xy} = -p h_{xy} - 2\eta \Gamma^{t}_{xy} + \ldots =$$

$$= -p h_{xy} - \eta \partial_t h_{xy} + O(\partial^2)$$

+ non-analyt. terms

With $h_{xy}(t) \sim e^{-i\omega t}$ (and $\bar{q}=0$):

$$\delta T_{xy} = -p h_{xy} + i\omega\eta h_{xy} + O(\omega^2) =$$

$$= -G^R_{T_{xy}T_{xy}}(\omega, \bar{0})\, h_{xy}$$

$$G^R_{T_{xy}T_{xy}}(\omega, 0) = p - i\omega\eta + O(\omega^2)$$

+ N/A terms

$$\boxed{\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \operatorname{Im} G^R_{T_{xy}T_{xy}}(\omega, \bar{q}=0)}$$

Kubo formula (Luttinger, 1963?).