M.Phys Option in Theoretical Physics: C6. Final Problem Sheet and Revision Questions, Quantum Field Theory

1.) (Derivative interactions)
Consider the theory of a real scalar field ϕ given by action

$$S = \int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{2}(\partial_\mu\phi)(\partial^\mu\phi)\phi,$$

By carrying out a derivation similar to the usual Feynman propagator, show that

$$\langle 0|T(\phi(x)(\partial_\mu\phi(y)))|0\rangle = \int \frac{d^4p}{(2\pi)^4}(ip_\mu)\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)} \ .$$

By applying the LSZ formula to

$$\left(\frac{i}{p_1^2 - m^2}\right)\left(\frac{i}{p_2^2 - m^2}\right)\left(\frac{i}{k^2 - m^2}\right)\langle \mathbf{p_1}, \mathbf{p_2}|iT|\mathbf{k}\rangle \ ,$$

and carrying out an expansion to order g, find the Feynman rule associated to the interaction vertex. You can assume that the $\partial_\mu\phi$ and $\partial^\mu\phi$ parts of the interaction are contracted with the fields corresponding to the *outgoing* particles.

(You may assume that Wick's theorem holds also for contractions of the form $\phi(x)\partial_\mu\phi(z)$ with the Feynman propagator modified to the expression of $\langle 0|T(\phi(x)(\partial_\mu\phi(y)))|0\rangle$ above.)

2.) (path integrals)
Consider the theory of a real scalar field ϕ given by action

$$S = \int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3,$$

a) Using the path integral formalism, check that the totally connected contribution to $G^{(3)}(x_1, x_2, x_3)$ to order g is the same as you would obtain in canonical quantization

b) Using the path integral formalism, derive an expression for the totally connected contribution to $G^{(4)}(x_1, x_2, x_3, x_4)$ to order g^2.

c) Draw Feynman diagrams and comment briefly on the relation of your results to the scattering process $\phi + \phi \to \phi + \phi$

3.) (Field theories with global $SO(n)$ symmetry)
a) The group $SO(n)$ consists of the real $n \times n$ matrices R satisfying $R^TR = \mathbf{1}_n$ and $\det(R) = 1$. Consider infinitesimal transformations of the form $R = \mathbf{1}_n + iT$ and determine the allowed matrices T, i.e. the Lie algebra of $SO(n)$. Show that the matrices T_{ab}, labelled by an antisymmetric pair of indices a, b and defined as $(T_{ab})_c{}^d = i(\delta_{ac}\delta_b^d - \delta_a^d\delta_{bc})$

(where $a, b, c, \ldots = 1, \ldots, n$) provide a basis for the Lie algebra. What is the dimension of this algebra?
b) A set of n scalar fields $\phi_a(x)$, where $a = 1, \ldots, n$, has a Lagrangian density $\mathcal{L} = \frac{1}{2}\sum_{a=1}^{n} \partial_\mu \phi_a \partial^\mu \phi_a - V$ with scalar potential $V = \frac{1}{2}m^2 \sum_{a=1}^{n} \phi_a^2 + \frac{\lambda}{4}(\sum_{a=1}^{n} \phi_a^2)^2$. Show that this Lagrangian density is invariant under $SO(n)$.
c) Compute the Noether currents of this $SO(n)$ symmetry.
d) For $m^2 < 0$ and $\lambda > 0$ find the minima of the scalar potential and show that the $SO(n)$ symmetry is broken to $SO(n-1)$. What is the number of Goldstone modes?

4.) (Interacting scalar field theory)
Two real scalar fields ϕ_1 and ϕ_2 with masses $m_1^2 > 0$ and $m_2^2 > 0$ transform as $\phi_1 \to -\phi_1$ and $\phi_2 \to \phi_2$ under a $\mathbb{Z}_2$ symmetry.
a) Write down the most general $\mathbb{Z}_2$ invariant Lagrangian (with standard kinetic terms and up to quartic terms in the fields) for ϕ_1 and ϕ_2.
b) Derive the Feynman rules for the vertices in this theory by computing the appropriate Green's functions.
c) Based on the results in b) and assuming negligible masses m_1 and m_2, find the amplitudes and differential cross sections for $\phi_1\phi_1 \to \phi_1\phi_1$ and $\phi_1\phi_1 \to \phi_2\phi_2$ scattering.

5.) (Scalar Yukawa theory)
The Lagrangian density of the scalar Yukawa theory is given by

$$\mathcal{L} = (\partial_\mu \varphi)^\dagger (\partial^\mu \varphi) - M^2 \varphi^\dagger \varphi + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - g\,\varphi^\dagger \varphi\,\phi\,,$$

where φ and ϕ is a complex and a real scalar, respectively. Assume that $g \ll M, m$.

a) The given Lagrangian density has an internal global symmetry. How does this symmetry act on the fields φ and ϕ? Calculate the associated Noether current J^μ and Noether charge Q.
b) Quantise the free theory ($g = 0$) and show that the normal-ordered Noether charge $:Q:$ can be written in the form $:Q:= N_+ - N_-$, where $N_\pm = \int d^3\tilde{p}\, a_\pm^\dagger(p) a_\pm(p)$ are the number operators of the "+" quanta (anti-nucleons, $\bar{N}$) and "−" quanta (nucleons, N) of the φ field.
c) Starting from the general result

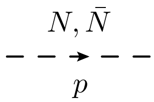
$$\sigma(p_1, p_2 \to q_1, q_2) = \frac{1}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \int \prod_{i=1,2} d^3\tilde{q}_i\, (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \times |\mathcal{M}(p_1, p_2 \to q_1, q_2)|^2\,,$$

evaluate the differential cross section for a generic $2 \to 2$ process. Work in the center-of-mass frame and assume for simplicity that the masses of all four external particles are equal to m. You should obtain

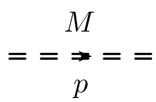
$$\frac{d\sigma(p_1, p_2 \to q_1, q_2)}{d\Omega} = \frac{1}{64\pi^2 E_{\text{CM}}^2} |\mathcal{M}(p_1, p_2 \to q_1, q_2)|^2\,,$$

where $d\Omega = d\phi\, d\cos\theta$ with $\theta \in [-\pi, \pi]$ and $\phi \in [0, 2\pi[$, while E_{CM} denotes the total center-of-mass energy.

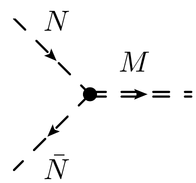
d) Draw the leading-order Feynman diagrams that describe the process $N\bar{N} \to MM$, where M denotes the quanta (mesons) of the ϕ field. Write down the matrix element $\mathcal{M}(N\bar{N} \to MM)$ using the momentum-space Feynman rules



$= \tilde{D}_F(p, M)$



$= \tilde{D}_F(p, m)$



$= -ig$

with $\tilde{D}_F(p, M) = i/(p^2 - M^2 + i\epsilon)$ and $\tilde{D}_F(p, m) = i/(p^2 - m^2 + i\epsilon)$. Employ Mandelstam variables. Using the results of (c) derive the differential cross section for $N\bar{N} \to MM$ production in the limit $E_{\rm CM} \gg M, m$ with $m = M$. Consider only the leading term in the expansion in powers of $m^2/E_{\rm CM}^2$.