M.Phys Option in Theoretical Physics: C6. Revision Problem Sheet 1

Question 1. Anharmonic Oscillator. Consider the anharmonic oscillator

$$H(\lambda) = \frac{\hat{p}^2}{2m} + \frac{\kappa}{2}\hat{x}^2 + \frac{\lambda}{4!}\hat{x}^4, \tag{1}$$

where $\kappa, \lambda > 0$.
(a) Discuss how the partition function $Z_\lambda(\beta)$ can be represented as a path integral. What kinds of paths are integrated over?
(b) What is the definition of the imaginary time Green's function of the harmonic oscillator ($\lambda = 0$)? How can it be expressed as a path integral (for $0 < \tau < \hbar\beta$)?
(c) Define a generating functional by

$$W_\lambda[J] = \mathcal{N} \int \mathcal{D}x(\tau)\; e^{\left\{\int_0^{\hbar\beta} d\tau \left[-\frac{1}{2}x(\tau)\hat{D}x(\tau) + J(\tau)x(\tau)\right] + U\left(x(\tau)\right)\right\}}\;, \tag{2}$$

where

$$U\big(x(\tau)\big) = -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \left[\frac{\lambda}{4!}x^4(\tau)\right], \quad \hat{D} = -\frac{m}{\hbar}\frac{d^2}{d\tau^2} + \frac{\kappa}{\hbar}. \tag{3}$$

Show that the partition function is equal to

$$Z_\lambda(\beta) = W_\lambda[0]. \tag{4}$$

(d) Show that the generating functional can be expressed in the form

$$W_\lambda[J] = \exp\left(U\Big(\frac{\delta}{\delta J(\tau)}\Big)\right) W_0[J]. \tag{5}$$

(e) Draw the Feynman diagrams for the first and second order perturbative corrections in λ to the partition function. Determine the first order corrections to the partition function.
(f) Determine the first order corrections to the two-point function

$$\langle T_\tau \bar{x}(\tau_1)\bar{x}(\tau_2)\rangle_\beta. \tag{6}$$

Draw the corresponding Feynman diagrams.

Question 2. The Hamiltonian for a one-dimensional ferromagnetic Ising model is

$$H = -J\sum_{n=1}^{L} S_n S_{n+1}\;,$$

where $J > 0$, S_n take the values $-2, -1, 0, 1, 2$, and where we have imposed periodic boundary conditions $S_{L+1} = S_1$. Write down the transfer matrix for this model. Give an exact express of the free energy per site in terms of the transfer matrix eigenvalues (do not attempt to calculate the eigenvalues explicitly). How does the answer simplify in the limit $L \to \infty$?

Derive an expression for the probability $\langle \delta_{S_1,1}\delta_{S_\ell,-1}\rangle$ that the spins at sites 1 and ℓ take the respective values 1 and -1 in terms of the transfer matrix eigenvalues λ_j and eigenvectors $|j\rangle$ (without calculating them).

Question 3. Consider the quantum spin-S chain

$$H = -\sum_{j=1}^{L} J^x S^x_j S^x_{j+1} + J^y S^y_j S^y_{j+1}\;, \tag{7}$$

where $J^x, J^y > 0$, $S^\alpha_{L+1} = S^\alpha_1$ and

$$\left(S^x_j\right)^2 + \left(S^y_j\right)^2 + \left(S^z_j\right)^2 = S(S+1). \tag{8}$$

(a) What are the classical ground states of this model (take the spins to be classical vectors of length S)? Consider the cases $J^x \neq J^y$ and $J^x = J^y$ separately.

(b) Explain the idea of spontaneous symmetry breaking for the cases $J^x > J^y$ and $J^x = J^y$ in this model. Which symmetries are broken?
(c) We now want to solve the model for $J^x > J^y$ in the linear spinwave approximation. To that end, we first carry out a rotation of spin quantization axis such that the classical ground state maps onto the ferromagnetic state, where all spins point along the z-axis in spin space. Show that this transformation is of the form

$$\tilde{S}^z_j = S^x_j\ ,\quad \tilde{S}^y_j = S^y_j\ ,\quad \tilde{S}^x_j = -S^z_j. \tag{9}$$

Express the Hamiltonian in terms of the rotated spin operators.

Then use the Holstein-Primakoff representation

$$\tilde{S}^z_j = S - a^\dagger_j a_j\ ,\quad \tilde{S}^+_j = \tilde{S}^x_j + i\tilde{S}^y_j = \sqrt{2S - a^\dagger_j a_j}\ a_j\ ,\quad [a_j, a^\dagger_k] = \delta_{j,k}\ ,\quad [a_j, a_k] = 0 \tag{10}$$

to carry out an expansion of H in inverse powers of S. Ignore the constant contribution and keep only the terms proportional to S. Give an explicit representation of the resulting Hamiltonian H_{LSW}, in terms of creation and annihilation operators in momentum space.
(d) What is a Bogoliubov transformation (BT), and what is its use? Diagonalize H_{LSW} by means of a BT (you may drop the constant contribution).

Question 4. A particle undergoes Brownian motion in one dimension. Its speed $v(t)$ satises the Langevin equation

$$\frac{dv(t)}{dt} = -\gamma v(t) + \eta(t), \tag{11}$$

where $\eta(t)$ is a Gaussian random variable, characterised by the averages

$$\langle \eta(t)\rangle = 0\ ,\quad \langle \eta(t_1)\eta(t_2)\rangle = \Gamma\ \delta(t_1 - t_2). \tag{12}$$

(a) Discuss the physical origin of each term in this equation.
(b) Show that, with initial condition $v(0) = 0$, the function

$$v_0(t) = \int_0^t dt'\ e^{-\gamma(t-t')}\eta(t') \tag{13}$$

is a solution to the Langevin equation for $t > 0$.
(c) Evaluate

$$\langle v_0(t_0)v_0(t_0 + t)\rangle \tag{14}$$

for $t_0 \to \infty$. Consider both cases $t < 0$ and $t > 0$. Explain how the result enables one to express Γ in terms of temperature and other parameters characterising the system.
(d) In the presence of an additional force on the particle, the Langevin equation has the modified form

$$\frac{dv(t)}{dt} = f - \gamma v(t) + \eta(t). \tag{15}$$

Show for constant f that the solution to this modified equation may be written in the form $v(t) = v_0(t) + u(t)$, and find $u(t)$.

Question 5. (a) Argue that the Landau free energy for a rotationally invariant system with a real, N-component order parameter $\vec{\phi}(\mathbf{r}) = (\phi_1(\mathbf{r}), \phi_2(\mathbf{r}), \ldots, \phi_N(\mathbf{r}))$ is of the form

$$F = \int \mathrm{d}^d\mathbf{r} \left[\frac{1}{2}|\nabla\vec{\phi}(\mathbf{r})|^2 + \alpha_2\vec{\phi}(\mathbf{r})\cdot\vec{\phi}(\mathbf{r}) + \alpha_4\left(\vec{\phi}(\mathbf{r})\cdot\vec{\phi}(\mathbf{r})\right)^2\right]\ . \tag{16}$$

(b) What can you say about the values of α_2 and α_4?
(c) Discuss the different phases and the nature of the phase transitions between them.
(d) Are there gapless Goldstone modes in the ordered phase (justify your answer)?