M.Phys Option in Theoretical Physics: C6. Revision Problem Sheet, Quantum Field Theory

1.) (Field theories with global $SO(n)$ symmetry)
a) The group $SO(n)$ consists of the real $n \times n$ matrices R satisfying $R^T R = \mathbf{1}_n$ and $\det(R) = 1$. Consider infinitesimal transformations of the form $R = \mathbf{1}_n + iT$ and determine the allowed matrices T, i.e. the Lie algebra of $SO(n)$. Show that the matrices T_{ab}, labelled by an antisymmetric pair of indices a, b and defined as $(T_{ab})_c{}^d = i(\delta_{ac}\delta_b^d - \delta_a^d\delta_{bc})$ (where $a, b, c, \ldots = 1, \ldots, n$) provide a basis for the Lie algebra. What is the dimension of this algebra?
b) A set of n scalar fields $\phi_a(x)$, where $a = 1, \ldots, n$, has a Lagrangian density $\mathcal{L} = \frac{1}{2}\sum_{a=1}^n \partial_\mu\phi_a\partial^\mu\phi_a - V$ with scalar potential $V = \frac{1}{2}m^2\sum_{a=1}^n \phi_a^2 + \frac{\lambda}{4}(\sum_{a=1}^n \phi_a^2)^2$. Show that this Lagrangian density is invariant under $SO(n)$.
c) Compute the Noether currents of this $SO(n)$ symmetry.
d) For $m^2 < 0$ and $\lambda > 0$ find the minima of the scalar potential and show that the $SO(n)$ symmetry is broken to $SO(n-1)$. What is the number of Goldstone modes?

2.) ($SU(3)$ gauge symmetry with spontaneous symmetry breaking)
The Lie group $SU(3)$ is defined as the set of complex 3×3 matrices U satisfying $U^\dagger U = \mathbf{1}_3$ and $\det(U) = 1$.
a) Show that the Lie algebra of $SU(3)$ consists of all hermitian 3×3 matrices with vanishing trace.
b) Write down a basis t^a for the Lie algebra of $SU(3)$ which contains the matrices ($i = 1, 2, 3$)

$$t^i = \begin{pmatrix} \tau^i & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tau^i \equiv \sigma_i/2$ with σ^i the usual Pauli matrices. What is the dimension of this $SU(3)$ Lie algebra?
c) Consider now an $SU(3)$ gauge theory with a scalar field ϕ in the *adjoint* representation. The covariant derivative of ϕ takes the form

$$D_\mu\phi_a = \partial_\mu\phi_a + g f_{abc} A_\mu^b \phi_c\,,$$

where A_μ^a denote the $SU(3)$ gauge fields and f_{abc} are the corresponding fully antisymmetric structure constants, i.e. $[t^a, t^b] = i f^{abc} t^c$. Which term in the Lagrangian give rise to the masses of the gauge fields?
d) The mass term can be written more clearly by defining the quantity $\Phi \equiv \phi_a t^a$. Use this notation together with the definition of the $SU(3)$ structure constants to write the mass term as

$$\mathcal{L} \supset -g^2\,\mathrm{tr}\left([t^a, \Phi][t^b, \Phi]\right) A_\mu^a A^{b\mu}\,.$$

e) Now let Φ acquire a vacuum expectation value (VEV). Consider the following two possibilities:

$$\langle\Phi\rangle = v\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad \langle\Phi\rangle = v\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Give the symmetry-breaking pattern induced by these VEVs and calculate the mass spectrum of the gauge bosons for both cases.

3.) (Interacting scalar field theory)
Two real scalar fields ϕ_1 and ϕ_2 with masses $m_1^2 > 0$ and $m_2^2 > 0$ transform as $\phi_1 \to -\phi_1$ and $\phi_2 \to \phi_2$ under a $\mathbb{Z}_2$ symmetry.
a) Write down the most general $\mathbb{Z}_2$ invariant Lagrangian (with standard kinetic terms and up to quartic terms in the fields) for ϕ_1 and ϕ_2.
b) Derive the Feynman rules for the vertices in this theory by computing the appropriate Green's functions.
c) Based on the results in b) and assuming negligible masses m_1 and m_2, find the amplitudes and differential cross sections for $\phi_1\phi_1 \to \phi_1\phi_1$ and $\phi_1\phi_1 \to \phi_2\phi_2$ scattering.

4.) (Scalar Yukawa theory)
The Lagrangian density of the scalar Yukawa theory is given by

$$\mathcal{L} = (\partial_\mu \varphi)^\dagger (\partial^\mu \varphi) - M^2 \varphi^\dagger \varphi + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - g\, \varphi^\dagger \varphi\, \phi\,,$$

where φ and ϕ is a complex and a real scalar, respectively. Assume that $g \ll M, m$.

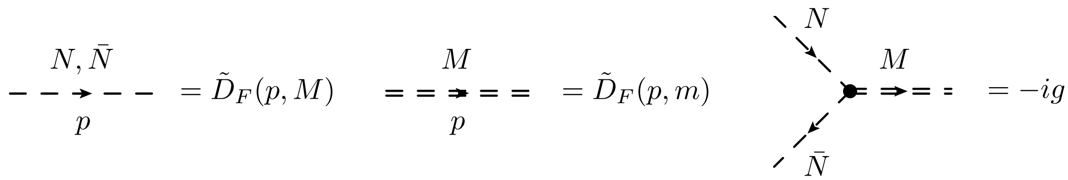
a) The given Lagrangian density has an internal global symmetry. How does this symmetry act on the fields φ and ϕ? Calculate the associated Noether current J^μ and Noether charge Q.
b) Quantise the free theory ($g = 0$) and show that the normal-ordered Noether charge $:Q:$ can be written in the form $:Q: = N_+ - N_-$, where $N_\pm = \int d^3\tilde{p}\, a_\pm^\dagger(p) a_\pm(p)$ are the number operators of the "+" quanta (anti-nucleons, $\bar{N}$) and "−" quanta (nucleons, N) of the φ field.
c) Starting from the general result

$$\sigma(p_1, p_2 \to q_1, q_2) = \frac{1}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}} \int \prod_{i=1,2} d^3\tilde{q}_i\, (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \times |\mathcal{M}(p_1, p_2 \to q_1, q_2)|^2\,,$$

evaluate the differential cross section for a generic $2 \to 2$ process. Work in the center-of-mass frame and assume for simplicity that the masses of all four external particles are equal to m. You should obtain

$$\frac{d\sigma(p_1, p_2 \to q_1, q_2)}{d\Omega} = \frac{1}{64\pi^2 E_{\text{CM}}^2} |\mathcal{M}(p_1, p_2 \to q_1, q_2)|^2\,,$$

where $d\Omega = d\phi\, d\cos\theta$ with $\theta \in [-\pi, \pi]$ and $\phi \in [0, 2\pi[$, while E_{CM} denotes the total center-of-mass energy.
d) Draw the leading-order Feynman diagrams that describe the process $N\bar{N} \to MM$, where M denotes the quanta (mesons) of the ϕ field. Write down the matrix element $\mathcal{M}(N\bar{N} \to MM)$ using the momentum-space Feynman rules



with $\tilde{D}_F(p, M) = i/(p^2 - M^2 + i\epsilon)$ and $\tilde{D}_F(p, m) = i/(p^2 - m^2 + i\epsilon)$. Employ Mandelstam variables. Using the results of (c) derive the differential cross section for $N\bar{N} \to MM$ production in the limit $E_{\mathrm{CM}} \gg M, m$ with $m = M$. Consider only the leading term in the expansion in powers of m^2/E_{CM}^2.