Quantum complex scalar field

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

$$(\Box + m^2)\varphi = 0 \qquad (\Box + m^2)\varphi^* = 0$$

$$\pi(x) = \dot{\varphi}^*$$

$$\mathcal{H} = \pi^* \pi + \nabla\varphi^* \nabla\varphi + m^2 \varphi^* \varphi$$

$$\varphi = (\phi_1 + i\phi_2)/\sqrt{2} \qquad \text{2 real d.o.f.}$$

$$\varphi \to \hat{\varphi}, \quad \varphi^* \to \hat{\varphi}^+$$

$$[\varphi(t,\bar{x}), \varphi(t,\bar{y})] = 0$$

$$[\varphi^+(t,\bar{x}), \varphi^+(t,\bar{y})] = 0$$

but

$$[\varphi(t,\bar{x}), \pi(t,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y})$$

$$[\varphi^+(t,\bar{x}), \pi^+(t,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y})$$

Solutions to KG eq:

$$\varphi(x) = \int d^3\tilde{p} \left(a_+(p) e^{-ipx} + a_-^+(p) e^{ipx} \right)$$

$$\varphi^+(x) = \int d^3\tilde{p} \left(a_- e^{-ipx} + a_+^+ e^{ipx} \right)$$

$$[a_\pm(p), a_\pm^+(q)] = (2\pi)^3 2\omega_p \delta^{(3)}(\bar{p}-\bar{q})$$

So we have 2 types of states:

$$|p,+\rangle = a_+^+ |0\rangle$$

$$|p,-\rangle = a_-^+ |0\rangle$$

Introduce:

$$N_\pm = \int d^3\tilde{p}\; a_\pm^+ a_\pm$$

For a <u>free</u> theory, $[N_\pm, H] = 0$

We have $U(1)$ symmetry in this theory and a Noether charge ($\Rightarrow$ operator)

$$Q = i\int d^3x \left(\varphi^+ \pi^+ - \pi\varphi\right)$$

<u>exercise</u>: show that (after normal ord.)

$$Q = \int d^3\tilde{p}\left(a_+^+ a_+ - a_-^+ a_-\right) = N_+ - N_-$$

$|p,+\rangle$: particles $\qquad [Q,H] = 0$
$|p,-\rangle$: antiparticles $\qquad$ in inter. theory as well.

Interacting Quantum Fields

* Self-interaction, e.g.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$(\Box + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \quad \text{Non-linear eom}$$

* Interaction among different fields

$$\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - M^2\varphi^*\varphi -$$
$$- \frac{1}{2}m^2\phi^2 - \underset{\sim\sim\sim}{g\varphi^*\varphi\phi}$$

Naive perturbative approach:

$$\mathcal{L} = \mathcal{L}_0 + \lambda\mathcal{L}_1$$

$\mathcal{L}_0$: free theory

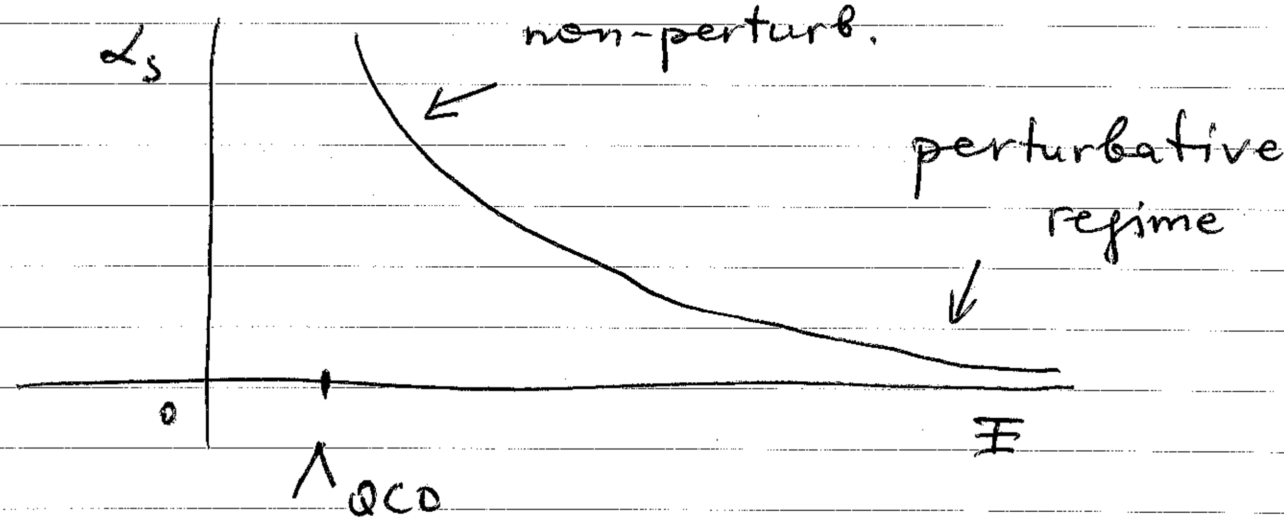
$\mathcal{L}_1$: perturbation „small" w.r.t. $\mathcal{L}_1$.

Criteria of „smallness"

- $\lambda \ll 1$ (?) But λ can be dimensionful..
- $\lambda \ll 1$, λ dimensionless (as in $\lambda\phi^4$ above)

Still problematic, since $\lambda = \lambda(E)$

E.g. in QCD $\alpha_s = g^2/4\pi$



* Non-perturbative methods in QFT are important (e.g. Lattice QCD); they are usually not universal, e.g. LQCD cannot adequately deal with time-dep. problems requiring Mink. space-time (only works with time-indep. situations)
* Perturbative approach gave many important physical predictions.

Interaction picture

Recall:

- Schrödinger picture
  - * states are time-dep. $|\varphi(\bar{x},t)\rangle_S$
  - * operators $\hat{O}_S$ are time-indep.

$$i\hbar\frac{\partial}{\partial t}|\varphi\rangle_S = \hat{H}|\varphi\rangle_S$$

$$|\varphi(\bar{x},t)\rangle_S = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\varphi(\bar{x},t_0)\rangle_S \equiv$$

$$\equiv \hat{U}(t,t_0)|\varphi(\bar{x},t_0)\rangle_S$$

<u>Physical quantities</u>:

$${}_S\langle\varphi(\bar{x},t)|\hat{O}_S|(\varphi(\bar{x},t)\rangle_S = \mathcal{O}(x,t)$$

- Heisenberg picture
  - * states are time-indep. $|\varphi\rangle_H$
  - * operators $\hat{O}_H$ are time-dep.

$$\langle\varphi(\bar{x},t)|\hat{O}_s|\varphi(\bar{x},t)\rangle_S = \langle\varphi(t_0)|\hat{O}_H|\varphi(t_0)\rangle,$$

where $\hat{O}_H = \hat{U}^+(t)\,\hat{O}_S\,\hat{U}(t)$.

e.o.m. $\frac{d}{dt}\hat{\mathcal{O}}_H = \frac{i}{\hbar}[\hat{H}, \hat{\mathcal{O}}_H]$

For $\hat{H} = \hat{H}_0 + \hat{H}_I$, <u>introduce</u>

$$\hat{\mathcal{O}}_{int.p.} = \hat{U}_0^+ \hat{\mathcal{O}}_S \hat{U}_0, \quad U_0 = e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$$

E.o.m. for $\hat{\mathcal{O}}_{int.p.}$ - ?

$$\dot{\mathcal{O}}_{int\,p} = \dot{U}_0^+ \mathcal{O}_S U_0 + U_0^+ \mathcal{O}_S \dot{U}_0 =$$

$$= \frac{i}{\hbar} H_0 \mathcal{O}_{int.p.} - \frac{i}{\hbar}\mathcal{O}_{int.p.} H_0 =$$

$$= \frac{i}{\hbar}[H_0, \mathcal{O}_{int.p.}]$$

<u>Moral</u>: $\hat{\mathcal{O}}_{int.p.}(t)$ e.o.m. is the same as in the Heis. picture for <u>free</u> fields

The states are not time-indep.:

$$|\varphi(t)\rangle_S = U_0(t)\,|\varphi(t)\rangle_{int.p.}$$

Since $i\hbar \frac{\partial |\varphi(t)\rangle_S}{\partial t} = (H_0 + H_I)|\varphi(t)\rangle_S$,

we find (check this !):

$$\boxed{\begin{array}{l} i\hbar \frac{\partial}{\partial t} |\varphi(t)\rangle_{int.p.} = \hat{H}_I^{int.p.} |\varphi(t)\rangle_{int.p.}, \\ \text{where } H_I^{int.p.} \equiv U_0^{+} H_I U_0 . \end{array}}$$

Since field operators $\hat{\phi}_{int.p}$ obey free e.o.m., their mode expansion and commut. rel. are the same as in free theory, e.g. for complex scalar field:

$$[\hat{\phi}_{int.p.}(x), \hat{\phi}^{+}_{int.p}(y)] = i\,\Delta(x-y)$$

Need to consider time evolution of the states $|\varphi(t)\rangle_{int.p.}$

Start at $t \to -\infty$ with some initial state $|i\rangle_{int.p.}$ It evolves to $|\varphi(+\infty)\rangle_{int.p.}$

$$|\varphi(+\infty)\rangle_{int.p.} = S|i\rangle$$

If $|\varphi(+\infty)\rangle_{int.p} = \sum_f \alpha_f |f\rangle$

for a complete set of final states, then

$$|\varphi(+\infty)\rangle_{int.p.} = \sum_f S_{fi} |f\rangle, \text{ where}$$

$$S_{fi} = \langle f | \varphi_{int.p.}(+\infty)\rangle = \langle f|S|i\rangle$$

Conservation of probability $\Rightarrow \sum_f |S_{fi}|^2 = 1$

Need to solve evol. eq. for $|\varphi(t)\rangle_{int.p.}$ with initial condition $|\varphi(-\infty)\rangle_{int.p} = |i\rangle$

Write the eq $i\hbar \frac{\partial}{\partial t}|\varphi(t)\rangle_{int.p} = \hat{H}_I^{int.p}|\varphi\rangle_{int.p.}$

as an integral eq:

$$|\varphi(t)\rangle_{in.p} = |i\rangle - \frac{i}{\hbar}\int_{-\infty}^{t} d\tau\, H_I^{in.p.}(\tau)|\varphi(\tau)\rangle_{in.p.}$$

Perturbative solution

$$|\varphi(t)\rangle_{int.p.} = \varphi_0 + \varphi_1 + \cdots$$

(Assuming a small param. in H_I such as $\lambda \ll 1$ or $\alpha_{em} = e^2/4\pi \sim 1/137$)

$$\varphi_0 = |i\rangle$$

$$\varphi_1 = -\frac{i}{\hbar} \int_{-\infty}^{t} d\tau_1 \, H_I^{int.p.}(\tau_1) \, |i\rangle$$

$$\varphi_2 = \left(-\frac{i}{\hbar}\right)^2 \int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \, H_I^{int.p.}(\tau_1) \, H_I^{int.p.}(\tau_2) \, |i\rangle$$

In the limit $t \to +\infty$;

$$S = \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_n \, H_I^{i.p.}(\tau_1) \cdots H_I^{i.p.}(\tau_n)$$

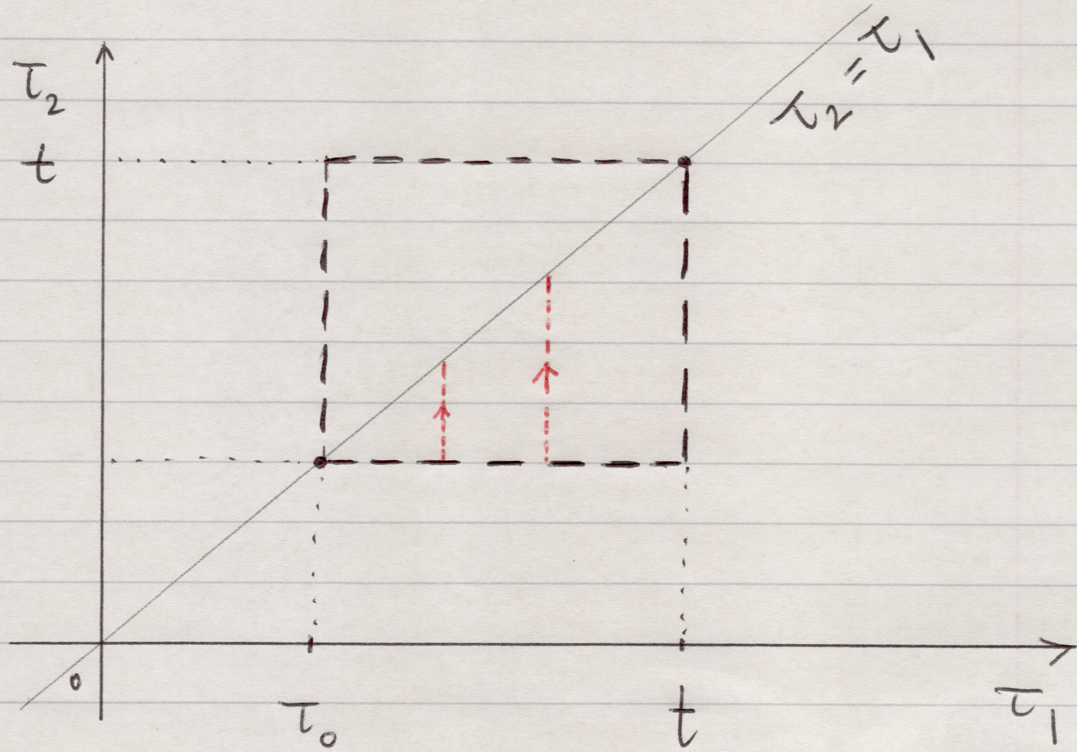
$$= \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \cdots \int_{-\infty}^{\infty} d\tau_n \, T\, H_I^{ip}(\tau_1) \cdots H_I^{ip}(\tau_n)$$

$$\equiv T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau \, H_I^{ip}(\tau)\right).$$

<u>Explicitly:</u> $(\tau_0 \to -\infty)$

$$\int_{\tau_0}^{t} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \, H_I^{int.p.}(\tau_1)\, H_I^{int.p.}(\tau_2) =$$

$$= \frac{1}{2} \int_{\tau_0}^{t} d\tau_1 \int_{\tau_0}^{t} d\tau_2 \; T\Big(H_I^{int.p.}(\tau_1)\, H_I^{int.p.}(\tau_2) \Big)$$



For Hamilt. density we have then (Dyson, 1949)

$$S = T \exp\left(-\frac{i}{\hbar} \int d^4x \, \mathcal{H}_I^{i.p.}(x) \right)$$

Need tools to deal with products

$$\int d^4x_1 \ldots \int d^4x_n \; T \, \mathcal{H}_I^{i.p.}(x_1) \ldots \mathcal{H}_I^{i.p.}(x_n)$$

$\Rightarrow$ Wick's theorem.