1) $m^2 > 0$: single min $\varphi_{min} = 0$

2) $m^2 < 0$: a circle of minima at

$$\varphi_{min} = \frac{v_0}{\sqrt{2}} e^{i\delta} , \qquad v_0 = \sqrt{\frac{-4m^2}{\lambda}}$$

The existence of the 1-dim. space of vacua is a consequence of the $U(1)$ symmetry of V.

Any specific choice of vac. breaks $U(1)$ spontaneously, e.g. with $\delta = 0$ we have $\varphi_{min} = v_0/\sqrt{2}$. Fluctuations:

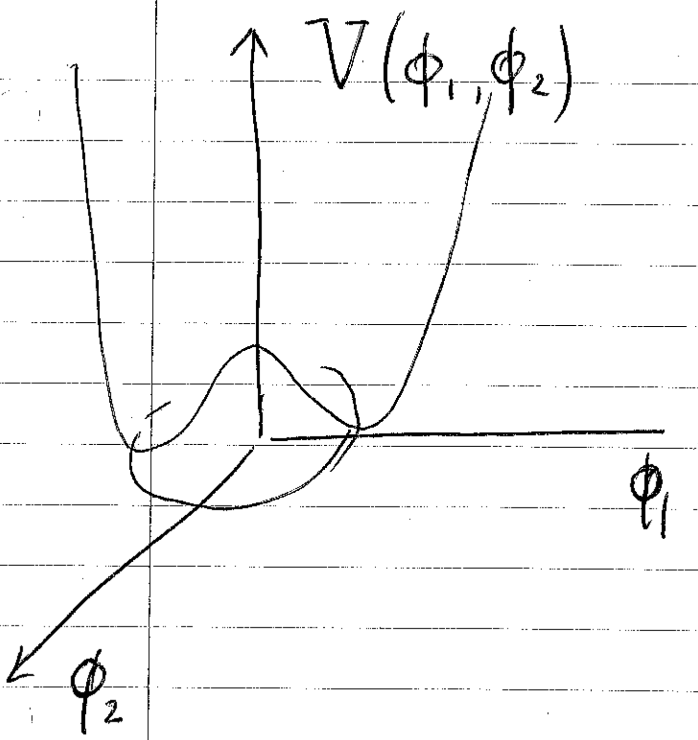
$$\varphi = \frac{1}{\sqrt{2}} \left(v_0 + \delta\varphi_1 + i\delta\varphi_2 \right)$$

$$\Rightarrow V = V_0 + \frac{1}{4} m^2 v_0^2 - m^2 \delta\varphi_1^2 + \mathcal{O}\left(\delta\varphi_1^3, \delta\varphi_1 \delta\varphi_2^2\right)$$

$\delta\varphi_1$: massive fluct. ($m^2 < 0$!)

$\delta\varphi_2$: massless. Goldstone

Spontaneously broken global symmetry
$\Rightarrow$ massless excitation (particle)



Nambu (1960)
Vaks and Larkin (1961)
Goldstone (1961)

Partial symmetry breaking: $SO(3)$ model

Consider real scalar fields φ^a, $a = 1,2,3$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi^a \partial^\mu \varphi^a - V(\varphi), \text{ where}$$

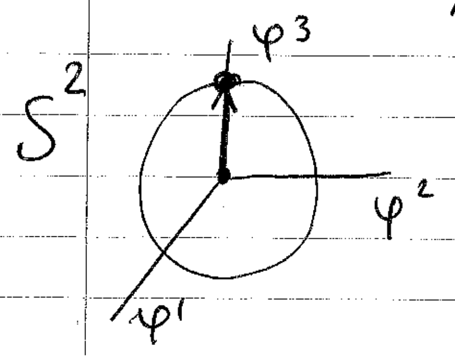
$$V(\varphi) = -\frac{\mu^2}{2}\varphi^a\varphi^a + \frac{\lambda}{4}(\varphi^a\varphi^a)^2 + \frac{\mu^4}{4\lambda}$$

$\mu^2 > 0$:

Vacua: min of $V(\varphi)$

$$\varphi^a\varphi^a = \varphi_0^2, \quad \varphi_0 = \mu/\sqrt{\lambda}.$$

Vacua are points of S^2: $(\varphi^1)^2 + (\varphi^2)^2 + (\varphi^3)^2 = \varphi_0^2$



Pick e.g. $\varphi^1 = 0$, $\varphi^2 = 0$
$\varphi^3 = \varphi_0$

But vacuum $\vec{\varphi}_0 = (0, 0, \varphi_0)$ is still invar. under $SO(2)$ rotations in the $\varphi^1 - \varphi^2$ plane: $\omega \vec{\varphi}_0 = \vec{\varphi}_0$, $\omega \in SO(2)$

Fluctuations around vac: $\omega \in SO(2) \subset SO(3)$

$$\varphi^1(x) = \theta^1(x) \quad \varphi^2(x) = \theta^2(x)$$

$$\varphi^3(x) = \varphi_0 + \chi(x)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \theta^1)^2 + \frac{1}{2}(\partial_\mu \theta^2)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - V,$$

$$V = -\frac{\mu^2}{2}\left[(\theta^1)^2 + (\theta^2)^2\right] - \frac{\mu^2}{2}(\varphi_0 + \chi)^2 + \frac{\lambda}{4}\left[(\theta^1)^2 + (\theta^2)^2 + (\varphi_0 + \chi)^2\right]^2 + \frac{\mu^4}{4\lambda}$$

Note: $\mathcal{L}$ is invar. under $SO(2)$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \theta^1)^2 + \frac{1}{2}(\partial_\mu \theta^2)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2$$

θ^1 and θ^2 are massless fields

SO(3) has 3 generators t_1, t_2, t_u

$$\omega = 1 + \varepsilon t_u \qquad \omega \in SO(2) \subset SO(3)$$

$$t_u \vec{\varphi}_0 = 0$$

$$\omega = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\alpha = \varepsilon \to 0$:

$$t_u = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

But $t_1 \bar{\varphi}_0 \neq 0$ and $t_2 \bar{\varphi}_0 \neq 0$

Let $\bar{n}_1 = t_1 \bar{\varphi}_0$, $\bar{n}_2 = t_2 \bar{\varphi}_0$

Then $\bar{\varphi} = \bar{\varphi}_0 + \tilde{\theta}_1 \bar{n}_1 + \tilde{\theta}_2 \bar{n}_2$

with $\tilde{\theta}_1, \tilde{\theta}_2 \ll 1$ is a vacuum:

$$\bar{\varphi} = (1 + \tilde{\theta}_1 t_1 + \tilde{\theta}_2 t_2)\bar{\varphi}_0 = \omega \bar{\varphi}_0$$

for ω close to $\mathbb{1}$.

$$V(\bar{\varphi}) = V(\bar{\varphi}_0) \qquad \left(\overset{\text{Note:}}{V(\bar{\varphi}_0) = 0}\right)$$

i.e. $V(\bar{\varphi}_0 + \tilde{\theta}_1 \bar{n}_1 + \tilde{\theta}_2 \bar{n}_2) = 0$

Since $\bar{\varphi}_0$ is a min of V $(V' = 0)$, the first non-trivial contrib. would be quadratic in $\tilde{\theta}_1, \tilde{\theta}_2$ but it vanishes $\Rightarrow$ $\bar{\varphi} = \bar{\varphi}_0 + \tilde{\theta}_1 \bar{n}_1 + \tilde{\theta}_2 \bar{n}_2$ is a massless fluctuation.

For $SO(3)$:

$$(t_a)_{bc} = \varepsilon_{abc}$$

$$\varphi_0^{\,a} = \delta^{a3} \varphi_0$$

$$(t_3)_{bc}\, \varphi_0^{\,c} = \varepsilon_{3bc}\, \delta^{c3} \varphi_0 = 0$$

$$n_1^{\,a} = (t_1)_{ab}\, \varphi_0^{\,b} = \varepsilon_{1ab}\, \delta^{b3} \varphi_0 = \delta^{a2} \varphi_0$$

$$n_2^{\,a} = \varepsilon_{2ab}\, \delta^{b3} \varphi_0 = -\delta^{a1} \varphi_0$$

And $\bar{\varphi}(x) = (-\tilde{\theta}_2 \varphi_0, \tilde{\theta}_1 \varphi_0, \varphi_0)$

i.e. $\theta^1 = -\tilde{\theta}_2 \varphi_0$ and $\theta^2 = \tilde{\theta}_1 \varphi_0$.

The moral: for each generator of a broken symmetry (or for each „broken" generator, if symmetry is partially broken) we get a massless field (Goldstone boson – in terms of particles).