$$\delta\varphi(x) = \varphi'(x') - \varphi(x) =$$

$$= \varphi'(x) + \underline{\frac{\partial\varphi'}{\partial x}}\,\delta x + O(\omega^2) - \varphi(x) =$$

$$= \varphi'(x) - \varphi(x) + \underline{\frac{\partial\varphi}{\partial x}}\,\delta x + O(\omega^2)$$

: differ by $O(\omega)$

$$= \bar{\delta}\varphi(x) + \frac{\partial\varphi}{\partial x}\,\delta x + O(\omega^2)$$

Therefore,

$$\delta S = \int d\Omega \left[\bar{\delta}\mathcal{L}(x) + \frac{\partial}{\partial x}\left(\mathcal{L}(x)\,\delta x\right)\right]$$

Now consider

$$\bar{\delta}\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\phi}\,\bar{\delta}\phi + \frac{\partial\mathcal{L}}{\partial(\partial\phi)}\,\bar{\delta}(\partial\phi) =$$

$$= \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi)}\right)\right]}_{=0 \ \underline{\text{on shell}}}\bar{\delta}\phi + \partial\left[\frac{\partial\mathcal{L}}{\partial\partial\phi}\,\bar{\delta}\phi\right]$$

$$\delta S = \int d\Omega\ \partial\left[\frac{\partial\mathcal{L}}{\partial\partial\phi}\,\bar{\delta}\phi + \mathcal{L}(x)\,\delta x\right] = 0$$

$$\delta S = \int d\Omega\, \partial^\mu j_\mu(x), \quad \text{where}$$

$$j_\mu = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\,\delta\phi - \left(\frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\frac{\partial \phi}{\partial x^\nu} - \eta_{\mu\nu}\mathcal{L}\right)\delta x^\nu$$

- Note: j_μ is not uniquely defined, can add $\forall\, F_\mu$ with $\partial_\mu F^\mu = 0$
- Recall $\begin{cases} \delta x^\nu = X^\nu_a\, \omega^a \\ \delta\phi = \phi'(x') - \phi(x) = \Psi_a(x)\,\omega^a \end{cases}$

$$\Rightarrow J^a_\mu = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\Psi_a - \left(\frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\frac{\partial \phi}{\partial x^\nu} - \eta_{\mu\nu}\mathcal{L}\right)X^\nu_a$$

(Noether currents)

$$Q_a = \int d^3x\, J^0_a \quad : \quad \dot{Q}_a = 0$$

Noether charges

Conserved <u>on shell</u>

Example: invariance under translations

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu$$

$\Rightarrow$ Noether current

$$\Theta_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \frac{\partial \phi}{\partial x^\nu} - \eta_{\mu\nu} \mathcal{L}$$

$\partial_\mu \Theta^{\mu\nu} = 0 \qquad \Theta^{\mu\nu}$ is known as the canonical energy-momentum tensor

Can always make it symmetric by

$$T_{\mu\nu} = \Theta_{\mu\nu} + \partial^\sigma \chi_{\sigma\mu\nu},$$

with $\forall\ \chi_{\sigma\mu\nu} = -\chi_{\mu\sigma\nu}$

$\partial_\mu T^{\mu\nu} = 0 \qquad$ (Belinfante, 1939)

This can coincide with another popular definition

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}.$$

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

Charges: $P^\nu = \int d^3x \, \Theta^{0\nu}$

$$P^0 = \int d^3x \, \Theta^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} \right) =$$

$$= \int d^3x \left(\pi \dot\phi - \mathcal{L} \right) = \int d^3x \, \mathcal{H}(x).$$

ex: $\Theta^{\mu\nu}$ and symmetric $T^{\mu\nu}$ lead to the same P^μ.

<u>Note:</u> $[T^{\mu\nu}] = D = 4$ in 4d.

Exercise (optional): show that

$$\bar\delta \phi_a = \{\phi_a, Q\}_{PB},$$

where $\{\ \}_{PB}$ is the Poisson bracket

$$\{\phi_a, Q\}_{PB} = \int d^3x' \left(\frac{\delta \phi_a(x)}{\delta \phi_b(x')} \frac{\delta Q}{\delta \pi_b(x')} - \frac{\delta \phi_a}{\delta \pi_b(x')} \frac{\delta Q}{\delta \phi_b(x')} \right).$$

Spontaneous symmetry breaking

<u>Example:</u> $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi)$

- e.o.m. $\partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \varphi} = 0$
- $P^0 = \int d^3x\, T^{00} = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}\partial_i \varphi \partial^i \varphi + V \right)$

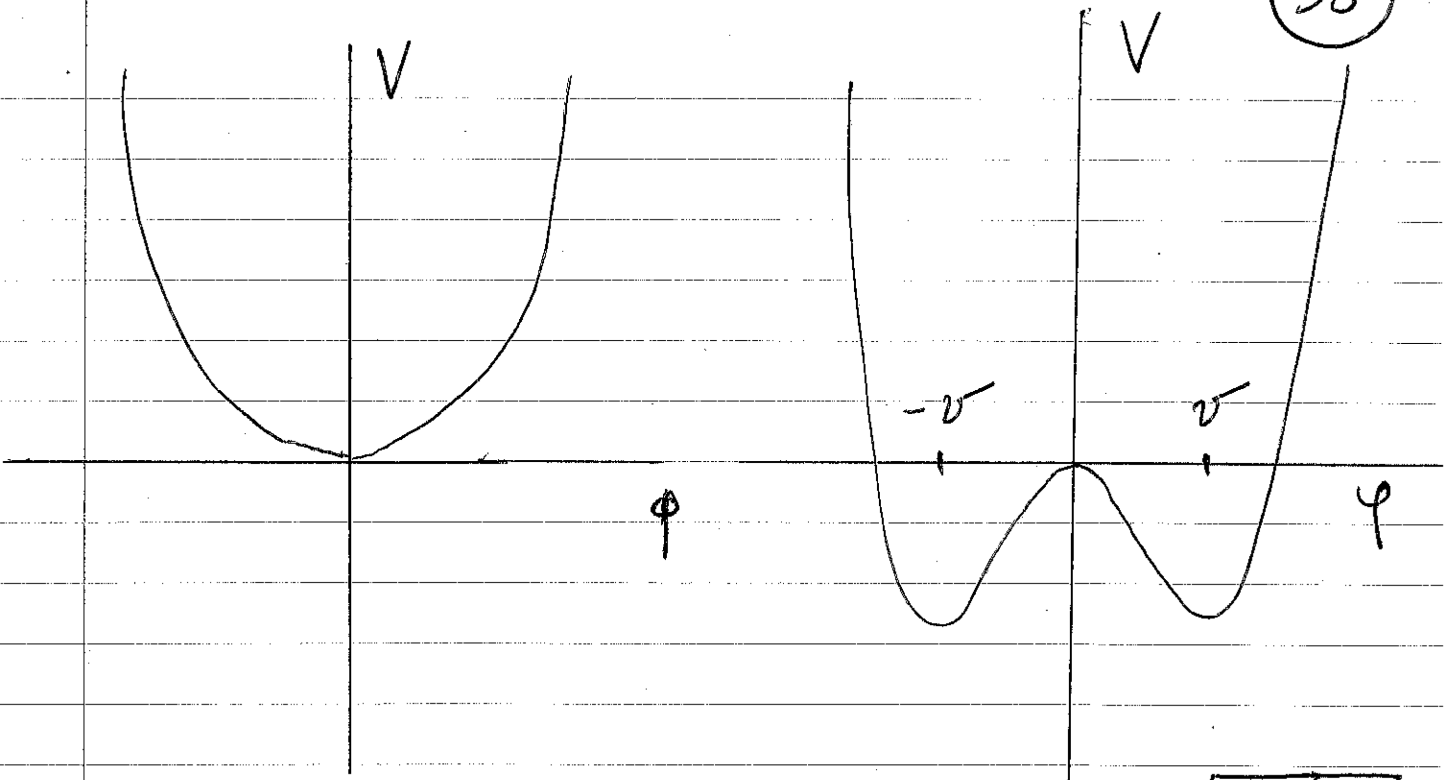
$$= \int d^3x\, \mathcal{H} = H \qquad \dot{H} = 0.$$

Here $\pi = \dot{\varphi}$.

Minima of V = minima of energy.
(vacua)

For $V(\phi) = V_0 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4$

- need $\lambda > 0$ to have energy bounded from below.
- depending on the sign of m^2, can have one min ($m^2 > 0$) or two ($m^2 < 0$)



$\varphi_{min} = 0$ $\qquad$ $\varphi_{min} = \pm\sqrt{\frac{-6m^2}{\lambda}} = \pm v$

$V_{min} = V_0$ $\qquad$ $V_{min} = V_0 - \frac{\lambda v^4}{24}$.

<u>Note</u>: $\mathcal{L}$ is invariant under <u>discrete</u> symmetry $\phi \to -\varphi$ ($\mathbb{Z}_2$ symmetry)

Vacua $\varphi_{min} = \pm v$ are interchanged under $\mathbb{Z}_2$. Choosing either solution breaks $\mathbb{Z}_2$ symmetry <u>spontaneously</u>

($\mathcal{L}$ is invar. but the solution to e.o.m. is not.)

<u>Example</u>: $\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi^* \varphi)$

$$V = V_0 + m^2 \varphi^* \varphi + \frac{\lambda}{4} (\varphi^* \varphi)^2$$

$$\varphi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2), \quad \phi_{1,2} : \text{real}$$

<u>Note</u>: $\mathcal{L}$ is invar. under $U(1)$

$$\begin{cases} \varphi \to \varphi' = e^{i\alpha} \varphi \\ \varphi^* \to \varphi^{*\prime} = e^{-i\alpha} \varphi^* \end{cases}$$

ex: find e.o.m.

ex: show that

$$\mathcal{H} = \pi \dot{\varphi} + \pi^* \dot{\varphi}^* = \pi^* \pi + \partial_i \varphi^* \partial^i \varphi + V(\varphi^* \varphi)$$

ex: find the $U(1)$ Noether current and charge. Show that

$$Q = i \int d^3x \, (\varphi^* \pi^* - \varphi \pi).$$