Canonical quantization

In classical mechanics (Hamiltonian formalism) functions $f(q,p)$ on phase space satisfy e.o.m.

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_a}\dot{q}_a + \frac{\partial f}{\partial p_a}\dot{p}_a$$

With Hamilton's eqs

$$\begin{cases} \dot{q}_a = \partial H / \partial p_a \\ \dot{p}_a = -\partial H / \partial q_a \end{cases}$$

this is

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_a}\frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial H}{\partial q_a} =$$

$$= \frac{\partial f}{\partial t} + \{f, H\}_{PB}$$

For $\forall$ A, B, C on phase space,

$$\{A, B\}_{PB} = -\{B, A\}_{PB}$$

$$\{A, \{BC\}\}_{PB} + \{C, \{A, B\}\}_{PB} + \{B, \{C, A\}\}_{PB} = 0$$

This is Lie algebra of functions on phase space with PB as Lie bracket.

Symmetries = canonical transformat. $(p, q) \to (P, Q)$ preserving PB.

Quantization: map this algebra of functions into algebra of operators on a Hilbert space with

$$[\hat{A}, \hat{B}] = i\hbar \{\hat{A}\, \hat{B}\}_{PB}$$

(Dirac, 1925)

Then $\quad [\hat{q}_a\ \hat{p}_b] = i\hbar\, \delta_{ab}$

$$[\hat{q}_a\ \hat{q}_b] = 0$$

$$[\hat{p}_a\ \hat{p}_b] = 0$$

$$\dot{\hat{q}}_a(t) = i\hbar^{-1}[\hat{H}, \hat{q}_a(t)] \qquad \dot{\hat{p}}_a(t) = i\hbar^{-1}[\hat{H}, \hat{p}_a(t)]$$

- Will drop $\hbar$ and hats from now on

Heisenberg picture: $A(t)$, states in Hilbert space are static.

<u>Note</u>: this map is not 1-1: quantum systems are more general than classical which only appear (uniquely) in the limit „$\hbar \to 0$" / i.e. one can „restore" classical system from quantum by taking the limit, but the opposite is not true.

In field theory:

$$[\phi_a(t,\bar{x}), \pi_b(t,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y})\delta_{ab}$$

$$[\phi_a(t,\bar{x})\ \phi_b(t,\bar{y})] = 0$$

$$[\pi_a(t,\bar{x})\ \pi_b(t,\bar{y})] = 0$$

$$\begin{cases} \dot{\phi}_a(t,\bar{x}) = i[H, \phi_a(t,\bar{x})] \\ \dot{\pi}_a(t,\bar{x}) = i[H, \pi_a(t,\bar{x})] \end{cases}$$

<u>Quantization of real scalar field</u>

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad \left(\begin{matrix}\text{free}\\ \text{field}\end{matrix}\right)$$

E.o.m. $(\Box + m^2)\phi = 0$

Classical solution is found via Fourier

$$\phi(t,\bar{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\bar{p}\bar{x}} \phi(t,\bar{p})$$

$$\Rightarrow \ddot{\phi}(t,\bar{p}) + \omega_{\bar{p}}^2 \phi(t,\bar{p}) = 0$$

$$\omega_{\bar{p}} = \sqrt{|\bar{p}|^2 + m^2}.$$

$$\phi(t,\bar{p}) = a_{\bar{p}}^{(1)} e^{-i\omega_{\bar{p}} t} + a_{\bar{p}}^{(2)} e^{i\omega_{\bar{p}} t}$$

Since $\phi(t,\bar{x})$ is real, $a_{\bar{p}}^{(2)} = a_{\bar{p}}^{(1)*}$

We can write $a_{\bar{p}}^{(1)} \equiv N_{\bar{p}} a_{\bar{p}}$ with a (real) normalisation $N_{\bar{p}}$ to be chosen later. Then

$$\phi(t,\bar{x}) = \int \frac{d^3p}{(2\pi)^3} N_{\bar{p}} \left(a_{\bar{p}} e^{-ipx} + a_{\bar{p}}^* e^{ipx} \right),$$

where $px = \omega_{\bar{p}} t - \bar{p}\bar{x} = p^0 x^0 - \bar{p}\bar{x}$.

$$\pi(t,\bar{x}) = \dot{\phi}(t,\bar{x}) = \int \frac{d^3q}{(2\pi)^3} N_{\bar{q}} \left(a_{\bar{q}} e^{-iqx} - a_{\bar{q}}^* e^{iqx} \right) \times (-i\omega_{\bar{q}}).$$

We now promote classical fields to (Hermitian) operators and impose

$$[\phi(t,\bar{x})\, \pi(t,\bar{y})] = i\,\delta^{(3)}(\bar{x}-\bar{y})$$

etc.

$$a_{\bar{p}},\ a_{\bar{p}}^{*} \rightarrow \hat{a}_{\bar{p}},\ \hat{a}_{\bar{p}}^{+}$$

<u>Exercise</u>: show that with $[\hat{a}_{\bar{p}}\, \hat{a}_{\bar{q}}^{+}] =$

$$= (2\pi)^3 2\omega_{\bar{p}}\, \delta^{(3)}(\bar{p}-\bar{q})$$

$[\hat{a}_{\bar{p}}, \hat{a}_{\bar{q}}] = 0, \quad [\hat{a}_{\bar{p}}^{+}, \hat{a}_{\bar{q}}^{+}] = 0$

and $N_{\bar{p}} = 1/2\omega_{\bar{p}}$ the correct commut. rel. for ϕ and π are recovered.

<u>Note</u>: $\delta(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha\tau}\, d\tau$.

<u>Note</u>: combination $2\omega_{\bar{p}}\, \delta^{(3)}(\bar{p}-\bar{q})$ is Lorentz-invar (see lecture notes).

<u>Now compute the Hamiltonian</u>

Classically, $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\phi\,\partial_i\phi + \frac{1}{2}m^2\phi^2$

$$H = \int d^3x \, \mathcal{H}$$

Exercise: show that

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\bar{p}}} \omega_{\bar{p}} \left(\hat{a}_p \hat{a}_p^+ + \hat{a}_p^+ \hat{a}_p \right) =$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2} a_{\bar{p}}^+ a_{\bar{p}} + (2\pi)^3 \frac{\omega_p}{2} \delta^{(3)}(0) \right)$$

Vacuum energy

Define vac. state by the condition

$$a_{\bar{p}} |0\rangle = 0 \qquad \forall \bar{p}$$

Then

$$H|0\rangle = E_0 |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} (2\pi)^3 \delta^{(3)}(0) |0\rangle$$

$$(2\pi)^3 \delta^{(3)}(0) = \lim_{L \to \infty} \int_{-L/2}^{L/2} d^3\bar{x} \, e^{i\bar{p}\bar{x}} \Big|_{\bar{p}=0} =$$

$$= \lim_{L \to \infty} L^3 = V \quad (3\text{-volume})$$

Define energy density $\varepsilon = E/V$

Then $\mathcal{E}_0 = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2}$

The integral is UV-divergent.
Introduce cutoff at energy scale E_*.
Then $\mathcal{E}_0 \sim E_*^4$ (note that $\mathcal{E}_0$ is energy density, i.e. $\mathcal{E}_0 \sim E/L^3 \sim M^4$)

Vac. energy couples to gravity:

$$\mathcal{L}_{grav} = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left(R - 2\underset{\substack{\uparrow \\ \text{cosmol. const}}}{\Lambda}\right)$$

C.c. term $\sim \frac{1}{8\pi G} \int d^4x \, \Lambda$

Note: $[\Lambda] = 1/L^2 \quad [G] = L^2$

$\Rightarrow \left[\frac{\Lambda}{8\pi G}\right] = M^4$: energy density ~~of~~ vac.

e.g. $\int \sqrt{-g} \, V_{min}(\phi) \, d^4x \sim \int d^4x \, \mathcal{E}_0^4$

$\Rightarrow \mathcal{E}_0 = \left(\Lambda/8\pi G\right)^{1/4} \sim 10^{-3} eV$

with $\Lambda \simeq 4.33 \cdot 10^{-66} eV^2$ (PDG)

$E_* \sim 10^{-3}\,eV$ (but QFT works fine at $E \sim 10\,TeV$) ?

On the other hand, imposing cutoff at $E_* \sim 10\,TeV$ or $E_P \sim 10^{19}\,GeV$ overpredicts Λ by orders of magnitude.

Solution for now: ignore gravity, measure the difference

$$:H: = H - E_0 = H - \langle 0|H|0\rangle$$

One can view this as the ordering problem: terms $p \cdot q$ and qp are classically equiv. but not in QM.

E.g. $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$ and

$\hat{H} = \frac{1}{2}(\omega\hat{q} - i\hat{p})(\omega\hat{q} + i\hat{p})$ are equiv. classically but in QM give

$$H = \omega(a^+ a + 1/2) \quad \text{and}$$

$$H = \omega a^+ a, \quad \text{resp.},$$

with the usual def. of a, a^+.

Normal ordering:

for $\phi \sim \hat{a} e^{-i\omega t} + \hat{a}^{+} e^{i\omega t}$

$$\sim \phi_+ + \phi_-$$

and $\chi \sim \chi_+ + \chi_-$ define

$$:\phi\chi: = \phi_-\chi_- + \phi_-\chi_+ + \chi_-\phi_+ + \phi_+\chi_+$$

so a^+ always stand to the left of a.

Then $:H: = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \omega_{\vec{p}} a^+_{\vec{p}} a_{\vec{p}}$

Note: the difference in vac. energy can be measured, e.g. in

- Casimir effect (see e.g. Itzykson-Zuber, Sect. 3.2.4)
- Spont. symm. breaking

• Single-particle states

$$|p\rangle = a^+_{\bar{p}} |0\rangle$$

We have

$$\langle p|q\rangle = \langle 0| a_{\bar{p}} a^+_{\bar{q}} |0\rangle =$$

$$= \langle 0| \big(\underbrace{a_{\bar{p}} a^+_{\bar{q}} - a^+_{\bar{q}} a_{\bar{p}}}_{[a_{\bar{p}} a^+_{\bar{q}}]}\big) |0\rangle = (2\pi)^3 2\omega_{\bar{p}} \delta^{(3)}(\bar{p}-\bar{q})$$

$$a_{\bar{p}} |0\rangle = 0$$

Note: $[H, a_{\bar{p}}] = -\omega_{\bar{p}}\, a_{\bar{p}}$

$$[H, a^+_{\bar{p}}] = \omega_{\bar{p}}\, a^+_{\bar{p}}$$

Here we use notations $:H: \equiv H$.

Then $\quad H|p\rangle = \omega_{\bar{p}} |p\rangle$

This quantum state corresp. to one real scalar with $E_p = \sqrt{\bar{p}^2 + m^2} = \omega_{\bar{p}}$.

• Recall that via Noether th. we can construct conserved charge $\bar{P}$ (momentum) for our $\mathcal{L}$.

Exercise: show that

$$\bar{P} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\bar{p}}} \, \bar{p} \, a^+_{\bar{p}} a_{\bar{p}}$$

is the operator of momentum with

$$\bar{P} \, |p\rangle = \bar{p} \, |p\rangle .$$

- Multi-particle states

$$|p_1, \dots p_n\rangle = a^+_{p_1} \dots a^+_{p_n} |0\rangle$$

– a state with n particles with momenta $p_1 \dots p_n$.

Note that $|p_1, p_2\rangle = a^+_{p_1} a^+_{p_2} |0\rangle =$
$= a^+_{p_2} a^+_{p_1} |0\rangle = |p_2, p_1\rangle$

i.e. the state is symmetric under exchange of particles (particles are ~~bosons~~) – this follows from

$$[a^+_p \, a^+_q] = 0$$ (specific type of commut. rel.)

Hilbert space: $|0\rangle$, $a^+_{\bar{p}}|0\rangle$, $a^+_{\bar{p}} a^+_{\bar{q}} |0\rangle$ ...

i.e. $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ Fock space

One can introduce the number operator

$$N = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} a^+_{\bar{p}} a_{\bar{p}}$$

Exercise: show $[N, a^+_{\bar{q}}] = a^+_{\bar{q}}$.

Show also that

$$N|p_1 \dots p_n\rangle = n|p_1 \cdots p_n\rangle$$

and $[N, H] = 0$

Note: $[N, H] = 0$ implies $\dot{N} = 0$

- the number of particles is conserved

This is true for free theories (no interact.)

but not in general!