Noether theorem

(global transform. of fields)

$$\mathcal{L} = \mathcal{L}(\phi^I, \partial_\mu \phi^I)$$

$\underline{I}$: collective index for indep. fields,

e.g. if $\mathcal{L} = \mathcal{L}(\varphi, \varphi^*, \partial_\mu \varphi, \partial_\mu \varphi^*)$,

then $\phi^1 = \varphi$, $\phi^2 = \varphi^*$.

$$\phi^I \to \phi'^I = \left(\delta^{IJ} + \varepsilon^a T_a^{IJ}\right)\phi^J -$$

- $\underline{\text{infinitesimal}}$ transf. of fields

parametrised by ε^a (ε^a are x-indep.)

$\underline{\text{Example}}$: $\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - V(\varphi^* \varphi)$

$$\begin{cases} \varphi \to \varphi' = e^{i\alpha}\varphi = (1 + i\alpha)\varphi + O(\alpha^2) \\ \varphi^* \to \varphi^{*\prime} = e^{-i\alpha}\varphi^* = (1 - i\alpha + O(\alpha^2))\varphi^* \end{cases}$$

Here $\varepsilon^a = \alpha$, $a = 1$.

$$T^{11} = i, \quad T^{22} = -i, \quad T^{12} = T^{21} = 0$$

$$T^{IJ} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Suppose $\mathcal{L}$ (and thus S) is invar. under $\delta\phi^I = \varepsilon^a T_a^{IJ}\phi^J$:

$$\delta\mathcal{L} = \mathcal{L}(\phi+\delta\phi,\ \partial_\mu\varphi + \delta\partial_\mu\varphi) - $$
$$- \mathcal{L}(\varphi,\ \partial_\mu\phi) = 0$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^I}\delta\phi^I + \frac{\partial\mathcal{L}}{\partial\phi^I_{,\mu}}\delta\partial_\mu\phi^I =$$

$$= \frac{\partial\mathcal{L}}{\partial\phi^I}\varepsilon^a T_a^{IJ}\phi^J + \frac{\partial\mathcal{L}}{\partial\phi^I_{,\mu}}\varepsilon^a T_a^{IJ}\partial_\mu\phi^J,$$

where $\phi^I_{,\mu} \equiv \partial_\mu\phi^I$.

Since parameters ε^a are arbitrary,

$$\frac{\partial\mathcal{L}}{\partial\phi^I} T_a^{IJ}\phi^J + \frac{\partial\mathcal{L}}{\partial\phi^J_{,\mu}} T_a^{IJ}\partial_\mu\phi^J = 0$$

Now, <u>on shell</u>: $\dfrac{\partial\mathcal{L}}{\partial\phi^I} = \partial_\mu\dfrac{\partial\mathcal{L}}{\partial\phi^I_{,\mu}}$

$$\Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi^I_{,\mu}} T_a^{IJ}\phi^J\right) = 0$$

i.e. $\partial_\mu J^\mu_a = 0$,

$$\boxed{J^\mu_a \equiv \frac{\partial \mathcal{L}}{\partial \phi^I_{,\mu}} T^{IJ}_a \phi^J}$$

Noether current: conserved locally <u>on shell</u> due to global symmetry of the action.

<u>Example</u>: complex scalar field

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} i \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} (-i) \varphi^*$$

$$= i\left(\partial^\mu \varphi^* \varphi - \partial^\mu \varphi \cdot \varphi^*\right)$$

$\partial_\mu J^\mu = 0$ (on shell) $\Rightarrow \dot{Q} = 0$,

where $Q = \int d^3x \, J^0$.

Noether theorem (1st)
(general case)

Suppose the action S is invar. under a global continuous symmetry. In general, both fields and coordinates are affected:

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + X^\mu_a(x)\,\omega^a & (*) \\ \phi^I(x) \to \phi'^I(x') = \phi^I(x) + \psi^I_a(x)\,\omega^a & (**) \end{cases}$$

- <u>global</u> transf. (param. ω^a are indep. of x)

- <u>continuous</u> (transf. belong to a Lie group => enough to consider infinitesimal ones).

To simplify things, we shall suppress all indices. We also introduce new notations.

$$\delta\phi(x) = \phi'(x') - \phi(x) = \psi(x)\,\omega$$

$$\bar{\delta}\phi(x) \equiv \phi'(x) - \phi(x)$$

We have $\bar{\delta}(\partial\phi) = \partial(\bar{\delta}\phi)$

<u>Indeed</u>, $\bar{\delta}(\partial\phi) = \dfrac{\partial\phi'(x)}{\partial x} - \dfrac{\partial\phi(x)}{\partial x} =$

$$= \frac{\partial}{\partial x}(\bar{\delta}\phi).$$

Note that $\partial(\delta\phi) \neq \delta(\partial\phi)$.

$$\frac{\partial}{\partial x}\delta\phi(x) = \frac{\partial}{\partial x}\Big(\phi'(x') - \phi(x)\Big) =$$

$$= \frac{\partial}{\partial x'}\phi'(x') - \frac{\partial\phi(x)}{\partial x} + \frac{\partial\phi'(x')}{\partial x} - \frac{\partial\phi'(x')}{\partial x'}$$

$$= \delta(\partial\phi) + \frac{\partial\phi'(x')}{\partial x} - \frac{\partial\phi'(x')}{\partial x'}$$

Now, $\phi'(x') = \phi(x) + \psi(x)\,\omega$, so

$$\frac{\partial}{\partial x}\phi'(x') = \frac{\partial\phi(x)}{\partial x} + \frac{\partial\psi}{\partial x}\cdot\omega$$

Also, $\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'} \frac{\partial}{\partial x} \Rightarrow$

$$\frac{\partial x'}{\partial x} = 1 + \frac{\partial X}{\partial x}\omega \Rightarrow \frac{\partial x}{\partial x'} = 1 - \frac{\partial X}{\partial x}\omega + O(\omega^2)$$

$$\Rightarrow \frac{\partial \phi'(x')}{\partial x'} = \left(1 - \frac{\partial X}{\partial x}\omega\right)\left(\frac{\partial \phi(x)}{\partial x} + \frac{\partial \psi}{\partial x}\omega\right) + O(\omega^2)$$

$$= \frac{\partial \phi(x)}{\partial x} + \frac{\partial \psi}{\partial x}\omega - \frac{\partial X}{\partial x}\frac{\partial \phi(x)}{\partial x}\omega + O(\omega^2)$$

Thus,

$$\frac{\partial}{\partial x}\delta\phi(x) = \delta\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi(x)}{\partial x}\frac{\partial X}{\partial x}\omega + O(\omega^2).$$

Consider $\delta S = \int_{V'} d\Omega' \mathcal{L}'(x') - \int_{V} d\Omega \mathcal{L}(x)$

$\delta S = 0$ Here $d\Omega = \sqrt{|g|}\, d^4x$

$$\mathcal{L}'(x') = \mathcal{L}\left(\phi'(x'), \frac{\partial \phi'}{\partial x'}(x')\right).$$

$$\delta S = \int_{V'} d\Omega' \, \delta \mathcal{L} + \int_{V'} d\Omega' \mathcal{L}(x) - \int_{V} d\Omega \, \mathcal{L}(x)$$

where $\delta \mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x)$.

$$d\Omega' = \left| \frac{\partial x'}{\partial x} \right| d\Omega = \begin{vmatrix} 1 + \frac{\partial \delta x^0}{\partial x^0} & \frac{\partial \delta x^0}{\partial x^1} & \cdots \\ \frac{\partial \delta x^1}{\partial x^0} & 1 + \frac{\partial \delta x^1}{\partial x^1} & \cdots \\ \vdots & & \end{vmatrix} \times d\Omega =$$

$$= \left(1 + \frac{\partial \delta x^\mu}{\partial x^\mu} + O(\omega^2) \right) d\Omega$$

(One can use e.g. log det = tr log)

So,

$$\delta S = \int_V d\Omega \left(\delta \mathcal{L}(x) + \frac{\partial \delta x}{\delta x} \mathcal{L}(x) \right) + O(\omega^2)$$

$$= \int_V d\Omega \left[\bar{\delta} \mathcal{L}(x) + \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \delta x}{\partial x} \mathcal{L} \right] + O(\omega^2)$$

Indeed, for $\forall$ scalar $\varphi(x)$: