Ex 6) Scalar electrodynamics

We can try to couple Maxwell field A^μ and charged scalar field:

$$\partial_\mu F^{\mu\nu} = e J^\nu(\varphi) \quad \left(\frac{\text{Note:}}{\partial_\nu J^\nu \equiv 0} \right)$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi - - e J_\mu A^\mu \quad (?)$$

<u>eom</u>: $-\partial_\mu \partial^\mu \varphi - m^2 \varphi + 2ie\, \partial_\mu \varphi A^\mu + + ie\varphi\, \partial_\mu A^\mu = 0$

$$-\partial_\mu \partial^\mu \varphi^* - m^2 \varphi^* - 2ie\, \partial_\mu \varphi^* A^\mu - - ie\varphi^*\, \partial_\mu A^\mu = 0$$

<u>Exercise</u>: derive these eoms.

Now, however, $\partial_\mu J^\mu \neq 0$ (on shell)!

$$\partial_\mu J^\mu = 2e\, \partial_\mu (\varphi^* \varphi A^\mu)$$

<u>Exercise</u>: show this (using eoms).

Modify J^μ: $J^\mu \to J^\mu_{old} - 2e\,\varphi^*\varphi A^\mu$

This new current is conserved.

The modified $\mathcal{L}$ is (show this):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D^\mu\varphi - m^2\varphi^*\varphi,$$

where $D_\mu\varphi = (\partial_\mu - ieA_\mu)\varphi$

$$(D_\mu\varphi)^* = (\partial_\mu + ieA_\mu)\varphi^*$$

<u>E.o.m.</u>

$$\begin{cases} D^\mu D_\mu\varphi + m^2\varphi = 0 \\ \partial_\mu F^{\mu\nu} = eJ^\nu \end{cases}$$

with $J^\nu = -i\left[\varphi^* D^\nu\varphi - (D^\nu\varphi)^*\varphi\right]$

<u>Exercise</u>: derive eom from $\mathcal{L}$.

Ex 7) Now add marginal term $\lambda(\varphi^*\varphi)^2$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D^\mu\varphi - m^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2$$

This is known as Abelian Higgs model

Comment on signs of m^2, λ.

From Lagrangian to Hamiltonian

For $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$, introduce momentum conjugate to ϕ_a by

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$$

and Hamiltonian density

$$\mathcal{H}(x) = \left(\pi_a(x)\,\dot{\phi}_a(x) - \mathcal{L}(\phi, \partial\phi)\right)\Big|_{\pi = \partial\mathcal{L}/\partial\dot{\phi}} \equiv \mathcal{H}(\pi, \phi).$$

The Hamiltonian is

$$H = \int d^3x\, \mathcal{H}(x).$$

Ex: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} \quad \Rightarrow$$

$$\mathcal{H} = \frac{1}{2}\pi^2(x) + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2$$

<u>Note</u>: transition from $(\phi, \dot{\phi})$ to (ϕ, π) via the Legendre transform

above is 1-1 if the Hessian

$$H_{ab} = \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}_a \partial \dot{\phi}_b}$$

is non-degenerate, i.e. if $\det H_{ab} \neq 0$.
If $\det H_{ab} = 0$, not all $\dot{\phi}$s can be expressed through πs. Such $\mathcal{L}$s are called singular (degenerate).
Corresponding theories have constraints.
Example: <u>all</u> gauge theories (QED, QCD etc, gravity).
Need special approach to quantize.

Symmetries and conservation laws.

Example

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi^* \varphi), \quad \text{e.g.}$$

$$V = m^2 \varphi^* \varphi + \lambda (\varphi^* \varphi)^2.$$

- $\mathcal{L}$ is Poincaré-invariant:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu$$

$$\varphi'(X') = \varphi(X)$$

$$\mathcal{L}'(X') = \mathcal{L}(X) \quad (\text{Lor. scalar})$$

The action is inv. $S' = \int_{V'} d^4x' \, \mathcal{L}'(x') =$

$$= \int_V d^4x \, \mathcal{L}(x)$$

Note: invar. measure $\int d^D x \sqrt{|g|} \, \mathcal{L}(x)$

where $g = \det g_{\mu\nu}$ for non-trivial metric.

- $\mathcal{L}$ is also invar. under

$$\varphi \to \varphi' = e^{i\alpha} \varphi \qquad \varphi^* \to \varphi^{*\prime} = e^{-i\alpha} \varphi^*$$

Global symmetry (α is indep. of x).

These are examples of continuous transf. and symmetries (parameter of transf. varies continuously) – typically, Lie groups.

- also important are discrete transf. (time reversal; parity)

Parity : $\begin{cases} \bar{x} \to -\bar{x} \\ t \to t \end{cases}$

Parity = reflection (e.g. $z \to -z$) plus rot. by 180° about a normal to the plane of reflection. A rotation-inv. theory is P-inv. iff it is reflection-invar.

Under parity: $\varphi(t, \bar{x}) \to \varphi(t, -\bar{x})$

or

$\varphi(t, \bar{x}) \to -\varphi(t, -\bar{x})$

(scalar or pseudoscalar)

$\mathcal{L}(t, \bar{x}) \to \mathcal{L}(t, -\bar{x})$ but the action remains invar.