Classical field theory

The example of electromagn. field suggests a strategy for quantization: start with a classical field theory, find the energy (Hamiltonian), then use the correspondence principle

$$[\hat{A}, \hat{B}] = i\hbar \{A, B\}_{PB}$$

to quantize the phase space.

Consider classical field theory of a system of fields $\phi_a(x)$ $a = 1, \cdots N$ with a Lagrangian density*

$$\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

The action

$$S(\Omega) = \int_\Omega d^4x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

Consider $\phi_a \to \phi_a + \delta\phi_a$ with

$\delta\phi_a(x) = 0$ on $\partial\Omega$ (Boundary of Ω)

* $L = \int d^3x \, \mathcal{L}$ is a Lagrangian

Var. principle $\delta S = 0$ gives e.o.m.

$$\delta S = \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right\}$$

$$= \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right\} \delta \phi_a +$$

$$+ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \Big|_{\partial \Omega} = 0$$

$\delta S = 0$

$\Rightarrow$ Euler-Lagrange e.o.m.

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0$$

Remark 1: $\mathcal{L}$ may contain higher derivatives (e.g. in GR). Then special care is required in writing the var. principle on manifolds with non-trivial boundary ($\partial\Omega \neq 0$), e.g. one may need to add extra terms to

the action to make var. principle well-defined (Gibbons-Hawking term in GR).
The canonical formalism is well known (Ostrogradski, 1850). Generically, it leads to pathologies on classical and quantum level (energy unbounded from below, ghosts etc, causality problems), unless the higher-deriv. terms are treated as small perturb. of the „Standard" dynamics given by the E-L eqs above.

Remark 2 We consider only local $\mathcal{L}$s - whose fields and derivatives are defined at the same point x. Non-local $\mathcal{L}$:

$$\mathcal{L}(x) = \int dy\, F(\phi(x), \phi(y)), \partial_x \phi, \partial_y \phi)$$

Non-loc. th. can be treated as loc. th. with infinite number of higher deriv.

Examples:

Ex 1) Maxwell theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

E-L eqs: $$\partial^\lambda \frac{\partial \mathcal{L}}{\partial(\partial^\lambda A^\varkappa)} = \frac{\partial \mathcal{L}}{\partial A^\varkappa}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\lambda A^\varkappa)} = \frac{\partial}{\partial(\partial^\lambda A^\varkappa)} \left[-\frac{1}{4} \eta_{\mu\rho} \eta_{\nu\sigma} F^{\rho\sigma} F^{\mu\nu} \right]$$

$$= -\frac{1}{4} \eta_{\mu\rho} \eta_{\nu\sigma} \left[\left(\delta^{\lambda\rho} \delta^{\varkappa\sigma} - \delta^{\lambda\sigma} \delta^{\varkappa\rho} \right) F^{\mu\nu} \right.$$

$$\left. + F^{\rho\sigma} \left(\delta^{\mu\lambda} \delta^{\nu\varkappa} - \delta^{\nu\lambda} \delta^{\mu\varkappa} \right) \right] =$$

$$= -\frac{1}{4} \left\{ \left(\eta_{\mu\lambda} \eta_{\nu\varkappa} - \eta_{\mu\varkappa} \eta_{\nu\lambda} \right) F^{\mu\nu} + \right.$$

$$\left. + F^{\rho\sigma} \left(\eta_{\lambda\rho} \eta_{\varkappa\sigma} - \eta_{\varkappa\rho} \eta_{\lambda\sigma} \right) \right\} =$$

$$= -\frac{1}{4} \left\{ F_{\lambda\varkappa} - F_{\varkappa\lambda} + F_{\lambda\varkappa} - F_{\varkappa\lambda} \right\} = -F_{\lambda\varkappa}.$$

$$\left.\begin{array}{l} LHS = -\partial^\lambda F_{\lambda\varkappa} \\ RHS = -J_\varkappa \end{array}\right\} \quad \boxed{\partial_\lambda F^{\lambda\varkappa} = J^\varkappa}$$

Ex2) Real scalar field (massive, free)

E.o.m. Lor.-inv. => $\mathcal{L}$ is a Lor. scalar

Linear e.o.m. => $\mathcal{L}$ quadratic in $\partial_\mu\phi$, ϕ

$$a\,\eta_{\mu\nu}\,\partial^\mu\phi\,\partial^\nu\phi$$

$$\cancel{\partial_\mu\phi\,\phi}$$

$$b\,\phi^2$$

$$\mathcal{L} = a\,\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + b\,\phi^2$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)} = a\,\eta^{\mu\nu}\left(\delta_{\mu\lambda}\,\partial_\nu\phi + \partial_\mu\phi\,\delta_{\nu\lambda}\right) =$$

$$= a\,\eta^{\lambda\nu}\partial_\nu\phi + a\,\eta^{\mu\lambda}\partial_\mu\phi = 2a\,\eta^{\lambda\mu}\partial_\mu\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = 2b\phi$$

=> e.o.m. $\boxed{\partial^\mu\partial_\mu\phi = \frac{b}{a}\phi}$

Can choose $a = 1/2$. Choose $b = -m^2/2$
(choice of sign is important)

$$\Rightarrow \quad \partial^\mu \partial_\mu \phi + m^2 \phi = 0$$

$$(\Box + m^2)\phi = 0 \quad \text{KG eq.}$$

Linear e.o.m.: if ϕ_1, ϕ_2 - solutions,
$\phi = \phi_1 + \phi_2$ is also a solution
(superposition principle - fields non-interacting - classical photons - ideal gas)

Ex 3) Interactions (sine-Gordon model)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right] =$$

$$= \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 + \frac{\lambda \phi^4}{4!} + O(\phi^6)$$

e.o.m. $\quad \Box\phi + \dfrac{m^3}{\sqrt{\lambda}} \sin \dfrac{\sqrt{\lambda}}{m}\phi = 0$

$$\Box\phi + m^2\phi - \lambda\phi^3 + \ldots = 0 \quad \text{(sine-Gordon eq)}$$