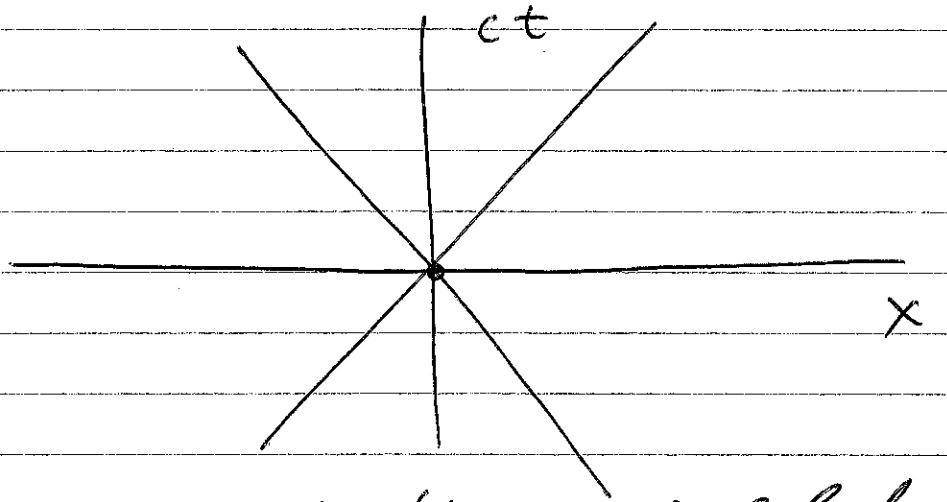
Causality

In SR, we have the light-cone



We can compute the probability for a particle in QM to travel from $\bar{x}=0$, $t=0$ to $(\bar{x}, t)$ outside the light-cone

$$\mathcal{P} = |\mathcal{A}|^2,$$

$$\mathcal{A} = \langle x | e^{-i\hat{H}t} | 0 \rangle =$$

$$= \int d^3_p \langle x | e^{-i\hat{H}t} | p \rangle \langle p | 0 \rangle$$

Recall: $\langle x | p \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\bar{p}\bar{x}}$

$$\mathcal{A} = \int d^3_p \frac{e^{i\bar{p}\bar{x}}}{(2\pi)^3} e^{-i\varepsilon_p t}$$

Here:
$\hat{H}|p\rangle = \varepsilon_p |p\rangle$
$\varepsilon_p = \sqrt{p^2 + m^2}$

(2)

$$\mathcal{A} = \frac{1}{2\pi^2 |\bar{x}|} \int_0^\infty dp\, p \sin p|\bar{x}|\, e^{-i\varepsilon_p t}$$

$$z = \sqrt{p^2 + m^2} \qquad dz = p\,dp / z$$

$$\Rightarrow \mathcal{A} = \frac{1}{2\pi^2 |\bar{x}|} \int_m^\infty dz \cdot z\, e^{-itz} \sin\left[|\bar{x}| \sqrt{z^2 - m^2}\right]$$

PBM – I – 2.5.42.3

$$\mathcal{A} = \frac{i m^2 t}{2\pi^2} \frac{K_2\left(m\sqrt{\bar{x}^2 - t^2}\right)}{\bar{x}^2 - t^2}$$

clearly <u>non-zero</u> for $t^2 - \bar{x}^2 < 0$ ...

Deep space-like region $|\bar{x}| \gg t$

$\mathcal{A} \sim e^{-m|\bar{x}|}$: small but finite.

This is problematic.

More generally, in QM, observables are Hermitian operators <u>not</u> attached to space-time points ⇒ this results in „action at a distance".

Observers separated by space-like intervals and measuring $\hat{O}_1, \hat{O}_2$ with $[\hat{O}_1, \hat{O}_2] \neq 0$ violate causality.

To avoid „action at a distance", promote $\hat{O}$ to a field $\hat{O}(x)$ - operator-valued (generalized) functions of space-time points x. Then causality is preserved if we demand

$$[\hat{O}_1(x_1), \hat{O}_2(x_2)] = 0$$

for space-like intervals, $(x_2 - x_1)^2 < 0$.

Electromagnetic field as Hamiltonian system

Maxwell eqs $\begin{cases} \partial_\mu F^{\mu\nu} = \mu_0 J^\nu \\ \partial_{[\mu} F_{\nu\lambda]} = 0 \quad \text{(Bianchi id)} \end{cases}$

constraints $\rightarrow$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu \rightarrow A_\mu - \partial_\mu f$

gauge invar.

Choose convenient gauge

e.g. $A^0 = 0$, $\operatorname{div} \bar{A} = 0$ $\Rightarrow$ eom

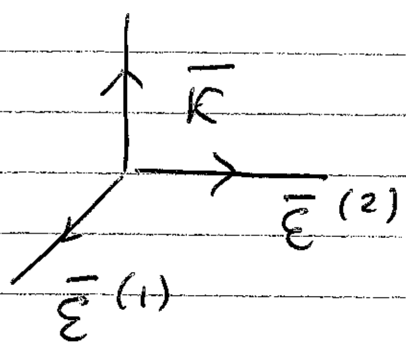
are simple: $\Box \bar{A} = 0$

Look for solutions in the form

$$A_i(t, \bar{x}) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3} \underbrace{N_k}_{\text{normalization}} \left(e^{i\bar{k}\bar{x}} \varepsilon_i^{(\lambda)}(k) A^{(\lambda)}(k,t) + \text{c.c.} \right)$$

Since $\partial_i A_i = 0 \Rightarrow \vec{k} \cdot \vec{\varepsilon}^{(\lambda)} = 0$: can

choose $\bar{\varepsilon}^{(\lambda)} \bar{\varepsilon}^{(\lambda')} = \delta^{\lambda\lambda'}$



Then $\Box A_i = 0$ gives

$$\ddot{A}^{(\lambda)}(k,t) + |\bar{k}|^2 c^2 A^{(\lambda)}(k,t) = 0$$

$$\Rightarrow A_i(t, \bar{x}) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3} \varepsilon_i^{(\lambda)} \left[a_\lambda(k) e^{-ikx} + a_\lambda^* e^{ikx} \right]$$

Here $kx = k^0 x^0 - \bar{k}\bar{x} = \omega_k t - \bar{k}\bar{x}$, $\omega_k = |\bar{k}| c$

The energy of the field

$$\mathcal{E} = \frac{1}{2}\int d^3x \left(\bar{E}^2 + \bar{B}^2\right) \quad \text{is}$$

$$\mathcal{E} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} \omega_k \, a_\lambda^*(k)\, a_\lambda(k)$$

with $N_k = c/\sqrt{2\omega_k}$.

Introducing

$$\begin{cases} Q_{\lambda,k} = \dfrac{a_\lambda + a_\lambda^*}{\sqrt{2\omega_k}} \\[2ex] P_{\lambda,k} = -\dfrac{i\omega_k (a_\lambda - a_\lambda^*)}{\sqrt{2\omega_k}} \end{cases}$$

we find

$$\mathcal{E} = \sum_\lambda \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left(P_{\lambda,k}^2 + \omega_k^2 Q_{\lambda,k}^2\right)$$

<u>Major discovery</u>: free electromagnetic field is an infinite collection of harmonic oscillators.

Canonical quantization: $[\hat{P}_{\lambda,k}\, \hat{Q}_{\lambda' k'}] = -i\hbar \delta_{\lambda\lambda'} \delta_{kk'}$

$[\hat{P}_{\lambda' k'}\, \hat{P}_{\lambda k}] = 0$, $[\hat{Q}_{\lambda k}\, \hat{Q}_{\lambda' k'}] = 0$