Generating functional for interacting fields

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = -V(\phi) \quad \left(e.g.\ V = \lambda\phi^4/4!\right)$$

$$W[J] = \mathcal{N}\int \mathcal{D}\phi\, \underbrace{e^{-i\int d^4x\, V(\phi)}}_{\text{can be expanded as polynomial in } \phi}\, e^{i\int d^4x(\mathcal{L}_0 + J\phi)}$$

Since $\frac{\delta}{\delta J(x)} e^{i\int d^4y\,(\mathcal{L}_0 + J(y)\phi(y))} =$

$$= i\,\phi(x)\, e^{i\int d^4y\,(\mathcal{L}_0 + J(y)\phi(y))}$$

we can write:

$$W[J] = \mathcal{N}\, e^{-i\int d^4x\, V\left(\frac{\delta}{i\,\delta J(x)}\right)}\, W_0[J]$$

$W[0] = 1 \quad \Rightarrow$

$$\mathcal{N}^{-1} = e^{-i\int d^4x\, V\left[\frac{\delta}{i\,\delta J(x)}\right]}\, W_0[J]\Big|_{J=0}$$

(Formal)
Perturbative expansion of $W[J]$
for $\lambda \ll 1$.

$$W[J] = W_0[J]\left(1 + \lambda w_1[J] + \lambda^2 w_2[J] + \cdots\right)$$

Remember that N also depends on λ;

$$N^{-1} \equiv W_0[J]\left(1 + \lambda u_1[J] + \lambda^2 u_2[J] + \cdots\right)\Big|_{J=0}$$

$$u_1[J] = W_0^{-1}[J]\left(-i\int d^4x\, V\left[\frac{\delta}{i\delta J}\right]\right) W_0[J]$$

$$u_2[J] = \cdots$$

$$\Rightarrow W[J] = W_0[J]\,\frac{1 + \lambda u_1[J] + \cdots}{1 + \lambda u_1[0] + \cdots}$$

$$\Rightarrow w_1[J] = u_1[J] - u_1[0]$$

$$w_2[J] = u_2[J] - u_2[0] - \left(u_1[J] - u_1[0]\right) u_1[0].$$

$\cdots$

Example: $V = \lambda \phi^4/4!$

$$u_1[J] = W_0^{-1}[J]\left[\frac{-i}{4!}\int d^4x \, \frac{1}{i^4}\left(\frac{\delta}{\delta J(x)}\right)^4\right]W_0[J]$$

$$W_0[J] = e^{-\frac{i}{2} J\Delta_F J}$$

$$J\Delta_F J \equiv \int d^4y\, d^4z\, J(y)\,\Delta_F(y-z)\,J(z).$$

- $$\frac{1}{i}\frac{\delta}{\delta J(x)}\, e^{-\frac{i}{2}J\Delta_F J} =$$

$$= -\left(\int d^4y\, \Delta_F(x-y)\,J(y)\right)e^{-\frac{i}{2}J\Delta_F J}$$

- $$\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^2 W_0[J] =$$

$$= \left[-\frac{1}{i}\Delta_F(0) + \left(\int d^4y\, \Delta_F(x-y)\,J(y)\right)^2\right]e^{-\frac{i}{2}J\Delta_F J}.$$

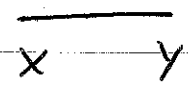
- $$\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3 e^{-\frac{i}{2}J\Delta_F J} = \left[3\,\frac{1}{i}\Delta_F(0)\int d^4y\,\Delta_F(x-y) \times J(y) - \left(\int d^4y\,\Delta_F(x-y)\,J(y)\right)^3\right]e^{-\frac{i}{2}J\Delta_F J}$$

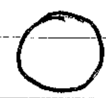
$$\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4 e^{-\frac{i}{2}J\Delta_F J} = \Big[3\,\frac{1}{i^2}\,\Delta_F^2(0) -$$

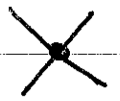
$$- 6\,\frac{1}{i}\,\Delta_F(0)\left(\int d^4y\, \Delta_F(x-y)\,J(y)\right)^2 +$$

$$+ \left(\int d^4y\, \Delta_F(x-y)\,J(y)\right)^4\Big]\, e^{-\frac{i}{2}J\Delta_F J}.$$

<u>Graphs:</u>

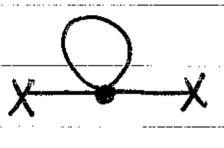
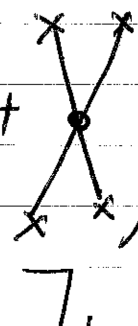
$i\,\Delta_F(x-y)$ 

$i\,\Delta_F(0)$ 

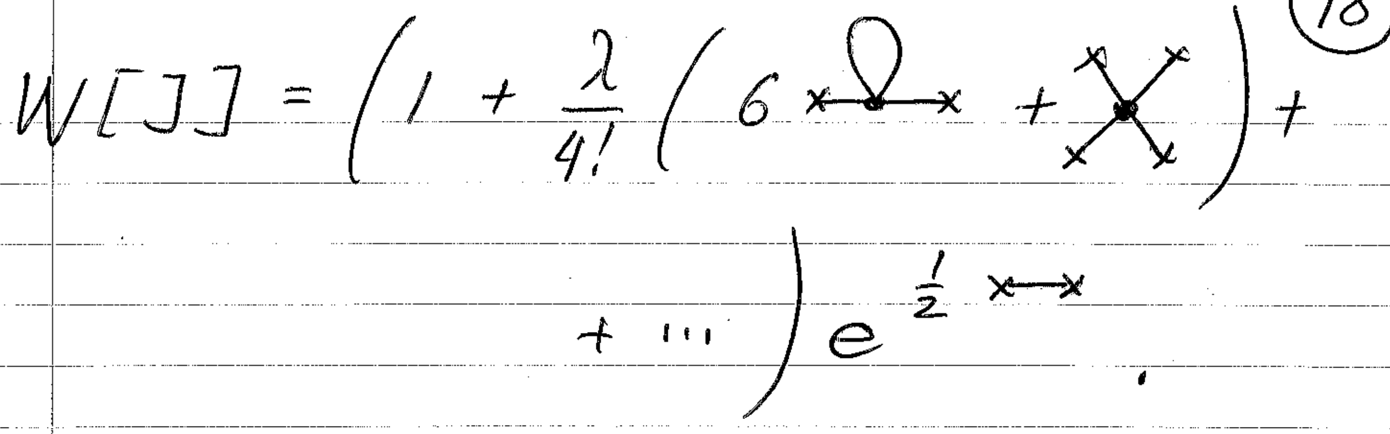
(λ is in front) $-i\int d^4x$ 

$i\int d^4x\, J(x)$ 

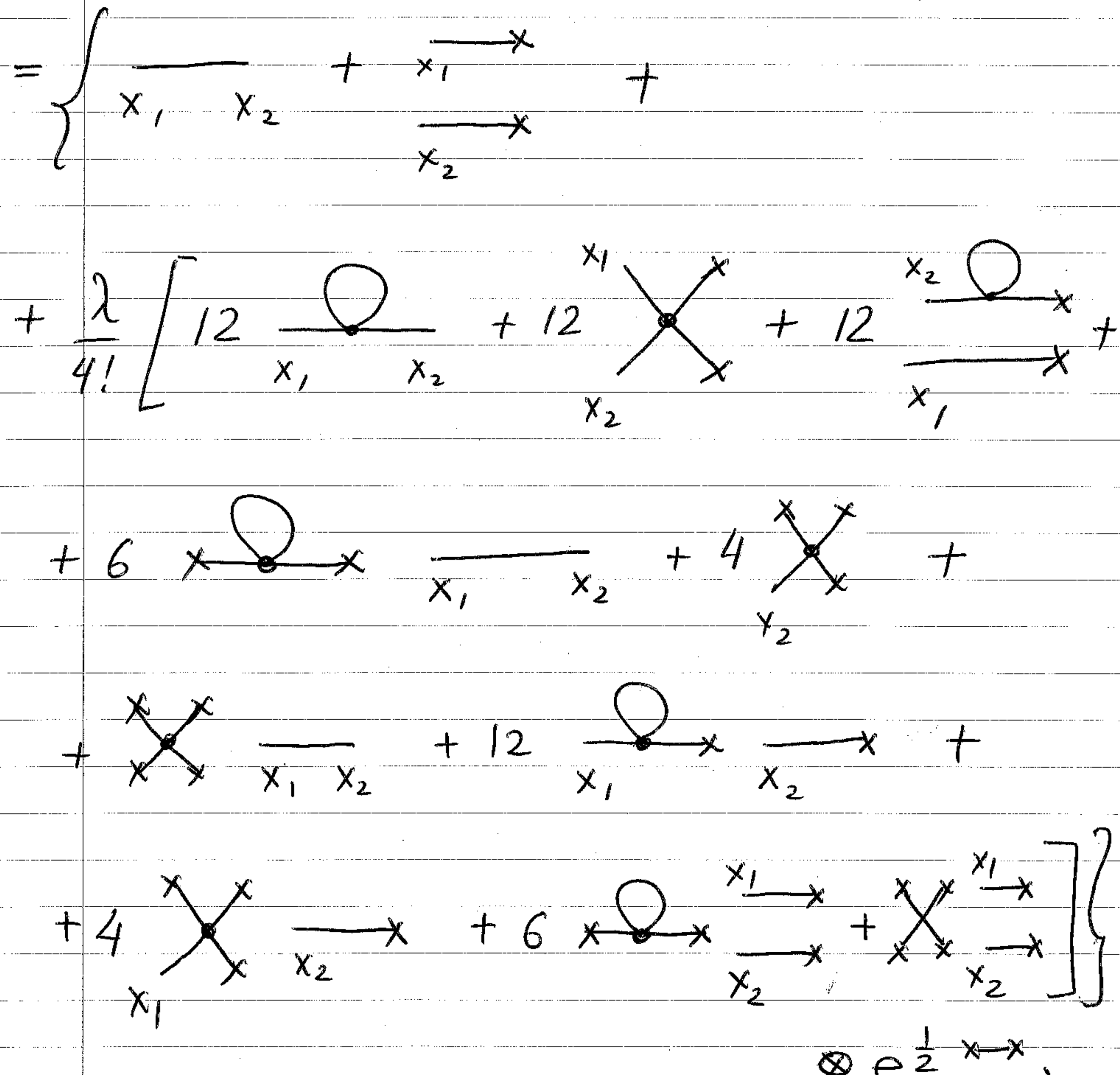
<u>Then:</u>

$u_1[J] = \frac{1}{4!}\Big(3$  $+ 6$  $+$  $\Big)$

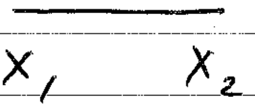
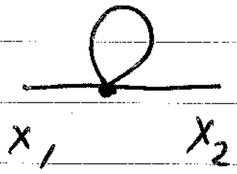
$$W[J] = \Big[1 + \lambda\left(u_1[J] - u_1[0]\right) + \dots\Big]\, W_0[J]$$



$$G^{(2)}(x_1, x_2) = \frac{1}{i^2} \frac{\delta W[J]}{\delta J(x_1)\, \delta J(x_2)} \Bigg|_{J=0} =$$



With $J=0$, all terms with $\rightarrow\!\times$ vanish.

$\Rightarrow G^{(2)}(x_1, x_2) =$  $+ \frac{\lambda}{2}$  $+$

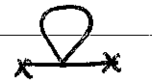
$+ O(\lambda^2)$

$$G^{(2)}(x_1, x_2) = i\Delta_F(x_1 - x_2) -$$

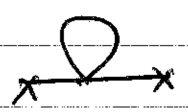
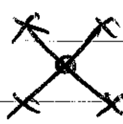
$$- \frac{\lambda}{2}\Delta_F(0)\int d^4x\, \Delta_F(x_1 - x)\Delta_F(x - x_2) +$$

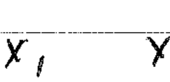
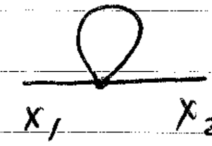
divergent! $\qquad + O(\lambda^2)$

Easier to deal with $Z[J] = -i \ln W[J]$

$iZ[J] = \ln\Big[\Big(1 + \frac{\lambda}{4!}\Big(6$  $+$  $\Big) + \cdots\Big) \times$

$\times\, e^{\frac{1}{2}\times\!-\!\times}\Big] =$

$= \frac{1}{2}\times\!\!-\!\!\times + \frac{\lambda}{4!}\Big(6$  $+$  $\Big) + O(\lambda^2)$

$G^{(2)}_{\text{connected}}(x_1, x_2) =$  $+ \frac{\lambda}{2}$  $+ O(\lambda^2)$

$= G^{(2)}(x_1, x_2)$

Effective action

Define $\phi_c(x) = \dfrac{\delta Z[J]}{\delta J(x)}$.

$$W[J] = e^{iZ[J]} = \frac{\int \mathcal{D}\phi \, e^{i\int d^4x (\mathcal{L}+J\phi)}}{\int \mathcal{D}\phi \, e^{i\int d^4x \, \mathcal{L}}}$$

$$\phi_c(x) = \frac{1}{iW[J]} \cdot \frac{\delta W[J]}{\delta J(x)} =$$

$$= \frac{\int \mathcal{D}\phi \, e^{i\int d^4y (\mathcal{L}+J\phi)} \phi(x)}{\int \mathcal{D}\phi \, e^{i\int d^4y (\mathcal{L}+J\phi)}} =$$

$$= \frac{\langle 0 | \hat{\phi}(x) | 0 \rangle_J}{\langle 0 | 0 \rangle_J}$$

$\phi_c(x)$: (normalised) vac. expect. value of $\hat{\phi}(x)$ in the presence of J (mean field)

Define effective action by

$$\Gamma(\phi_c) = Z[J] - \int d^4x\, J(x)\, \phi_c(x)$$

$$\frac{\delta \Gamma(\phi_c)}{\delta J(y)} = \phi_c(y) - \phi_c(y) = 0$$

$\Rightarrow \Gamma(\phi_c)$ is indep. of J.

- Compute $\phi_c(x)$ for free fields

$$Z_0[J] = -i \ln W_0[J] = -\frac{1}{2} \int d^4x\, d^4y\, J(x) \Delta_F(x-y) \times J(y).$$

$$\Rightarrow \phi_c(x) = -\int d^4y\, \Delta_F(x-y)\, J(y).$$

i.e. $\phi_c(x)$ is a solution to

$$(\Box + m^2)\, \phi_c(x) = J(x)$$

- recall that $(\Box + m^2)\, \Delta_F(x-y) = -\delta^{(4)}(x-y)$

i.e. Δ_F is a Green's function for $\Box + m^2$.

$$\Gamma_0[\phi_c] = Z_0[J] - \int d^4x\, J(x)\phi_c(x) =$$

$$= \frac{1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y).$$

Now subst. $J(x) = (\Box + m^2)\phi_c(x)$ and integrate by parts (exercise!)

$$\Rightarrow \Gamma_0[\phi_c] = \frac{1}{2}\int d^4x\, (\partial_\mu\phi_c\partial^\mu\phi_c - m^2\phi_c^2)$$

For interacting theory will also get terms involving $\int d^4x_1 \cdots d^4x_n\, \underbrace{\phi_c(x_1)\cdots\phi_c(x_n)}_{\text{non-local}}$

Expanding all $\phi_c(x_k) = \phi_c(x) + \partial\phi_c(x) + \dots$

get <u>local</u> effective action with higher deriv.

$$\Gamma[\phi_c] = \int d^4x\, \Big(-U(\phi_c) + \frac{1}{2}\partial_\mu\phi_c\partial^\mu\phi^c F(\phi^c) + \text{higher deriv.}\Big)$$

$U(\phi_c) = \frac{m^2}{2}\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 +$ quantum corrections expressed via ϕ_c

$\Rightarrow$ effective potential.