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Real scalar field in Euclidean space $\mathbb{R}^4$

Recall: $t_{Mink} = -i\tau_E$

$c\tau_E \equiv x_E^0 \qquad ds^2_{Mink} = c^2 dt^2 - d\bar{x}^2 \rightarrow$

$$\rightarrow ds_E^2 = -\left(dx_E^{0\,2} + d\bar{x}_E^2\right)$$

$$\mathcal{L}_E = -\frac{1}{2}\frac{\partial\phi}{\partial x_E^\mu}\frac{\partial\phi}{\partial x_E^\mu} - \frac{m^2}{2}\phi^2 + \mathcal{L}_{int}$$

$$W_E[J] = N_E \int \mathcal{D}\phi \, e^{\int d^4x_E \left(\mathcal{L}_E + J(x_E)\phi(x_E)\right)}$$

- Consider first $\mathcal{L}_{int} = 0$, $W_E = W_E^0$:

$$W_E^0[J] = N_E \int \mathcal{D}\phi \, e^{-\int d^4x_E \left[\frac{1}{2}\partial^E_\mu\phi\,\partial^E_\mu\phi + \frac{m^2}{2}\phi^2 - J\phi\right]}$$

This is a Gaussian integral:

$$\int dx_1 \cdots dx_n \, e^{-\frac{1}{2}\sum_{ij} x_i K_{ij} x_j + \sum_k J_k x_k} =$$

$$= (\det K)^{-1/2} \, e^{\frac{1}{2}\sum_{ij} J_i (K^{-1})_{ij} J_j}$$

Here the finite-dim scalar product is replaced with $(J, \phi) = \int d^4 x_E \, J(x_E) \phi(x_E)$

The term $\partial_\mu^E \phi \, \partial_\mu^E \phi$ can be integrated by parts (with assumption that fields vanish at ∞). Example in 1 dim:

$$\int dx \, \frac{\partial \phi(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} = \int dx \, dy \, \frac{\partial \phi(x)}{\partial x} \frac{\partial \phi(y)}{\partial y} \delta(x-y)$$

$$= -\int dx \, dy \, \phi(y) \frac{\partial}{\partial y} \left(\frac{\partial \phi(x)}{\partial x} \delta(x-y) \right).$$

$$\Rightarrow \int d^4 x_E \, \partial_\mu^E \phi \, \partial_\mu^E \phi = -\int d^4 x_E \, d^4 y_E \, \times$$

$$\times \, \phi(y_E) \underbrace{\frac{\partial}{\partial y_E^\mu} \frac{\partial}{\partial x_E^\mu}}_{} \delta^{(4)}(x_E - y_E) \phi(x_E)$$

$$W_E^0[J] = N_E \int \mathcal{D}\phi \, e^{-\frac{1}{2} \int d^4 x_E \, d^4 y_E \, \phi(y_E) K(y_E, x_E) \phi(x_E) + \int d^4 x_E \, J(x_E) \phi(x_E)}$$

$$K = (-\Box_E + m^2) \, \delta^{(4)}(x_E - y_E).$$

We need to compute K^{-1}: then

$$W_E^{\circ}[J] = N_E (\det K)^{-1/2} e^{\frac{1}{2}\int d^4x_E\, d^4y_E\, J(y_E) K^{-1}(y_E, x_E) J(x_E)}$$

Also, $W_E^{\circ}[0] = N_E (\det K)^{-1/2}$; setting

$$\Rightarrow W_E^{\circ}[0] = 1 \Rightarrow N_E = (\det K)^{1/2}$$

$$\Rightarrow W_E^{\circ}[J] = e^{\frac{1}{2}\int d^4x_E\, d^4y_E\, J(y_E) K^{-1}(y_E, x_E) J(x_E)}$$

Note: $K(y_E, x_E) = \left(\frac{\partial}{\partial y_E^{\mu}}\frac{\partial}{\partial x_E^{\mu}} + m^2\right)\int \frac{d^4p_E}{(2\pi)^4} e^{-ip_E(y_E - x_E)}$

$$= \int \frac{d^4p_E}{(2\pi)^4}(p_E^2 + m^2) e^{-ip_E(y_E - x_E)}$$

$$\int d^4x_E\, K^{-1}(y_E, x_E)\, K(x_E, z_E) = \delta^{(4)}(y_E - z_E)$$

$$K^{-1}(p_E) = \frac{1}{p_E^2 + m^2}$$

$$K^{-1}(y, x) = \int \frac{d^4p_E}{(2\pi)^4} e^{-ip_E(y_E - x_E)} \frac{1}{p_E^2 + m^2}$$

$K^{-1} = \Delta_F^E$ is a Green's function:

$$(-\Box_E + m^2)\,\Delta_F^E(y-x) = \delta^{(4)}(y-x).$$

<u>Note</u>: integrating over p_0^E now does not encounter poles (they are now at $p_0^E = \pm i\sqrt{\bar{p}^2+m^2}$).

$$W_0^E[J] = e^{\frac{1}{2}\int d^4x_E\, d^4y_E\, J(y_E)\,\Delta_F^E(y-x)\,J(x_E)}$$

Analytic continuation to Mink:

$$\Delta_F^E(i(y_0-x_0),\, \bar{y}'-\bar{x}') = \left\{\begin{array}{l} p_E^0 = -i\,p^0_{MINK} \\ \bar{p}_E = \bar{p}_{MINK} \end{array}\right\}$$

$$= i\int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \frac{1}{p^2-m^2} = i\,\Delta_F(y-x).$$

$$\boxed{W_0[J] = e^{-\frac{i}{2}\int d^4x\, d^4y\, J(y)\,\Delta_F(y-x)\,J(x)}}$$

$$G_0^{(N)}(x_1 \cdots x_N) = \frac{1}{i^N} \left. \frac{\delta^N W_0[J]}{\delta J(x_1) \cdots \delta J(x_N)} \right|_{J=0}$$

<u>$N=1$:</u>

$$G_0^{(1)}(x_1) = \frac{1}{i} \left. \frac{\delta W_0[J]}{\delta J(x_1)} \right|_{J=0} =$$

$$= \frac{1}{i}\left(-\frac{i}{2}\right) 2 \int d^4x \; J(x)\, \Delta_F(x-x_1)\, e^{-\frac{i}{2} J \Delta J}$$

With $J=0 \Rightarrow G_0^{(1)}(x_1) = 0$.

<u>$N=2$:</u>

$$G_0^{(2)}(x_1, x_2) = \frac{1}{i^2} \left. \frac{\delta^2 W_0[J]}{\delta J(x_1)\, \delta J(x_2)} \right|_{J=0} =$$

$$= i\, \Delta_F(x_2 - x_1)$$

More explicitly:

$$\frac{1}{i^2} \frac{\delta^2 W_0[J]}{\delta J(x_1)\, \delta J(x_2)} = i \Big[\Big(\Delta_F(x_2 - x_1) \; -$$

$$- i \int d^4x \, J(x) \, \Delta_F(x - x_1) \int d^4x \, J(x) \, \Delta_F(x - x_2) \Big)$$

$$\times e^{-\frac{i}{2} J \Delta J} \Big] .$$

Clearly, $\displaystyle \frac{\delta^3 W_0[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3)} \Bigg|_{J=0} = 0$

Explicitly:

$$\frac{1}{i^3} \frac{\delta^3 W_0[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3)} = \Big[- i \Delta_F(x_1 - x_3) \int d^4x \, J(x) \, \Delta_F(x - x_3)$$

$$- i \Delta_F(x_1 - x_3) \int d^4x \, J(x) \Delta_F(x - x_2) -$$

$$- i \Delta_F(x_2 - x_3) \int d^4x \, J(x) \, \Delta_F(x - x_1) -$$

$$- \int d^4x \, J(x) \, \Delta_F(x - x_1) \int d^4x \, J(x) \, \Delta_F(x - x_2) \times$$

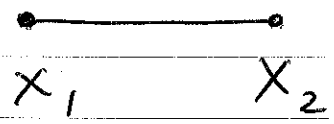
$$\times \int d^4x \, J(x) \, \Delta_F(x - x_3) \Big] e^{-\frac{i}{2} J \Delta J} .$$

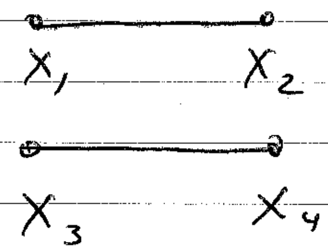
Continuing to $G_0^{(4)}(x_1, x_2, x_3, x_4)$, we obtain (check this!):

$$G_0^{(4)}(x_1 x_2 x_3 x_4) = i^2 \Big[\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4)$$

$$+ \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3) \Big]$$

Feynman diagrams:

$G_0^{(2)}(x_1, x_2) =$ 

$G_0^{(4)}(x_1, x_2, x_3, x_4) =$ 

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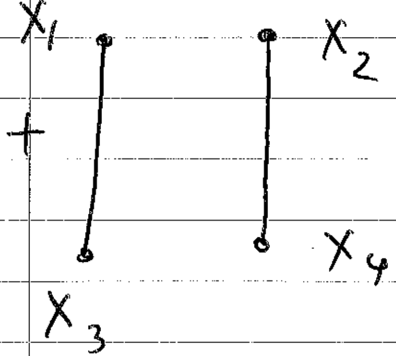
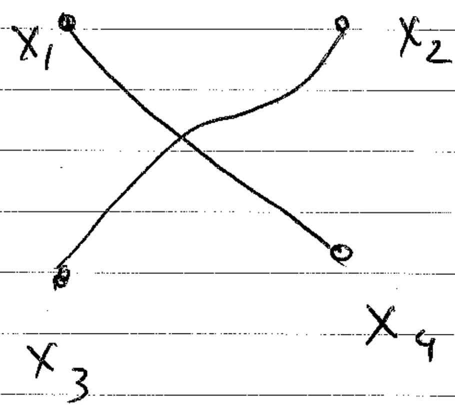
$+$ 

$+$ 

$$\Rightarrow \quad G_0^{(2n+1)}(x_1 \dots x_{2n+1}) = 0$$

$$G_0^{(2n)}(x_1 \dots x_{2n}) = \sum_{perm} G_0^{(2)}(x_{k_1}, x_{k_2}) \cdots G_0^{(2)}(x_{k_{2n-1}}, x_{k_{2n}})$$

e.g. $G_0^{(6)}$ has 15 terms.

What about $Z[J]$ - ?

$$W[J] = e^{iZ[J]}$$

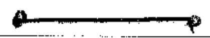
Note: $W[0] = 1 \Rightarrow Z[0] = 0$.

$$G^{(N)}_{\text{connected}} = \frac{1}{i^{N-1}} \left. \frac{\delta^N Z[J]}{\delta J(x_1) \cdots \delta J(x_N)} \right|_{J=0}$$

Since $W_0[J] = e^{-\frac{i}{2}\int d^4x\, d^4y\, J(y)\Delta_F(y-x) J(x)}$

$$Z_0[J] = -\frac{1}{2}\int d^4x\, d^4y\, J(y)\Delta_F(y-x) J(x)$$

$$G^{(2)}_{0\,\text{connected}}(x_1, x_2) = i\Delta_F(x_1 - x_2) = G_0^{(2)}(x_1 - x_2)$$

$=$ 

x_1 x_2

All other $G^{(n)}_{0,\text{connected}}$ vanish since $Z_0[J]$ is only quadratic in J.