# Path Integrals in QFT

In QFT, we are interested in $N$-point functions

$$G^{(N)}(x_1 \cdots x_N) = \langle 0 | T\, \hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_N) | 0 \rangle$$

Recall that in QM:

$$\langle x_f | e^{-i\hat{H}(t_f - t_i)/\hbar} | x_i \rangle =$$

$$= N \int_{\substack{x(t_i) = x_i \\ x(t_f) = x_f}} \mathcal{D}x(t)\, e^{\frac{i}{\hbar} \int_{t_i}^{t_f} \mathcal{L}(x, \dot{x})\, dt}$$

Here $|x_i\rangle$ and $|x_f\rangle$ are eigenstates of the position operator. The amplitude is the position space repres. of the Schröd. evolution operator.

The analog of $G^{(2)}(x_1, x_2)$ in QM is

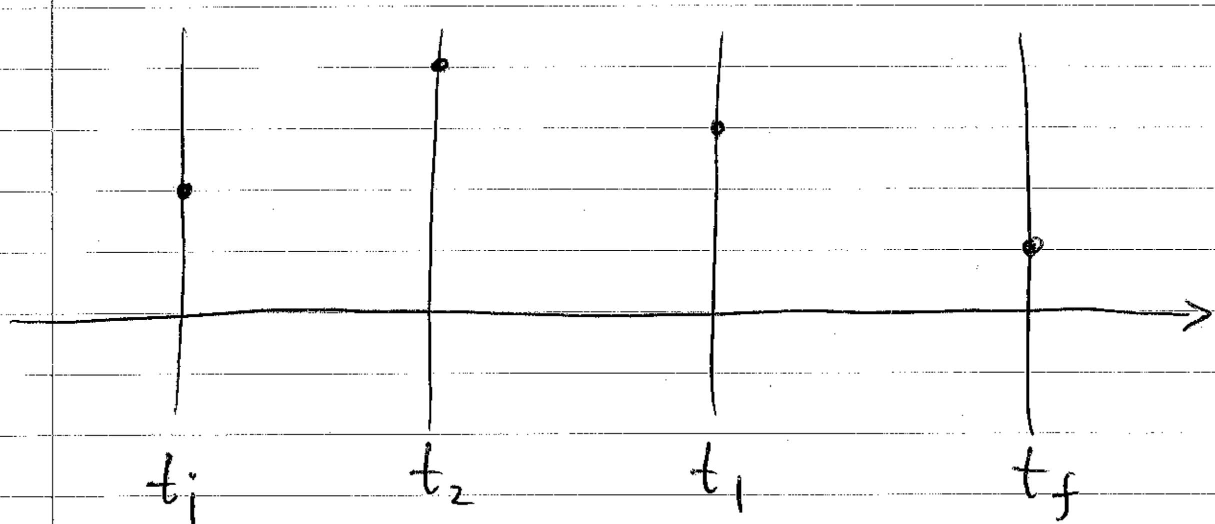
$$\langle x_f\, t_f | T\, \hat{x}(t_1)\, \hat{x}(t_2) | x_i\, t_i \rangle =$$

$$= \langle x_f | e^{-i\hat{H}(t_f - t_1)} \hat{X}_s e^{-i\hat{H}(t_1 - t_2)} \hat{X}_s e^{-i\hat{H}(t_2 - t_i)} | x_i \rangle$$

$$= \int dx_1 dx_2 \langle x_f | e^{-i\hat{H}(t_f - t_1)} | x_1 \rangle \langle x_1 | \hat{X}_s e^{-i\hat{H}(t_1 - t_2)} | x_2 \rangle$$

$$\times \langle x_2 | \hat{X}_s e^{-i\hat{H}(t_2 - t_i)} | x_i \rangle$$

Here $t_1 > t_2$




$x(t_2) = x_2 \qquad x(t_1) = x_1$

E.g. $\langle x_f | e^{-iH(t_f - t_1)} | x_1 \rangle =$

$$= N \int \mathcal{D}x(t)\, e^{i \int_{t_1}^{t_f} \mathcal{L}\, dt}$$

$x(t_1) = x_1$
$x(t_f) = x_f$

Integration over $x_1$, $x_2$ removes constraints $x(t_1) = x_1$, $x(t_2) = x_2$

$$\Rightarrow \langle x_f\, t_f \,|\, T\hat{x}(t_1)\,\hat{x}(t_2)\,|\, x_i\, t_i \rangle =$$

$$= \mathcal{N}\int \mathcal{D}x(t)\; x(t_1)\,x(t_2)\, e^{i\int_{t_i}^{t_f}\mathcal{L}\,dt}$$

$x(t_i) = x_i$
$x(t_f) = x_f$

(Same for $t_1 < t_2$.)

We also need to relate $|x_{i,f}\, t_{i,f}\rangle$ to $|0\rangle$ at $t \to \mp\infty$. This gives an overall phase (see Peskin - Schröder, Sect. 4.2)

$$\Rightarrow \langle 0|T\hat{x}(t_1)\cdots\hat{x}(t_N)|0\rangle =$$

$$= \frac{\int \mathcal{D}x(t)\, e^{iS[x]}\, x(t_1)\,x(t_2)\cdots x(t_N)}{\int \mathcal{D}x(t)\, e^{iS[x]}}$$

In QFT:

$$\langle 0|T\hat{\phi}_H(x_1)\cdots\hat{\phi}_H(x_N)|0\rangle =$$

$$= \mathcal{N}\int \mathcal{D}\phi\; \phi(x_1)\cdots\phi(x_N)\, e^{iS[\phi]}$$

Introduce the generating functional

$$W[J] = \mathcal{N} \int \mathcal{D}\phi \, e^{i\int d^4x \left[\mathcal{L} + J(x)\phi(x)\right]}$$

Then

$$G^{(N)}(x_1 \cdots x_N) = \frac{1}{i^N} \frac{\delta^N W[J]}{\delta J(x_1) \cdots \delta J(x_N)}\Bigg|_{J=0}$$

$$\left( W[J]\Big|_{J=0} = W[0] = \langle 0|0\rangle = 1 \quad \Rightarrow \mathcal{N} = \frac{1}{\int \mathcal{D}\phi\, e^{iS}} \right)$$

It is convenient to use

$$Z[J] = -i \ln W[J]$$

$$\Rightarrow G^{(N)}_{\text{connected}}(x_1 \cdots x_N) = \frac{1}{i^{n-1}} \frac{\delta^N Z[J]}{\delta J(x_1) \cdots \delta J(x_N)}\Bigg|_{J=0}$$

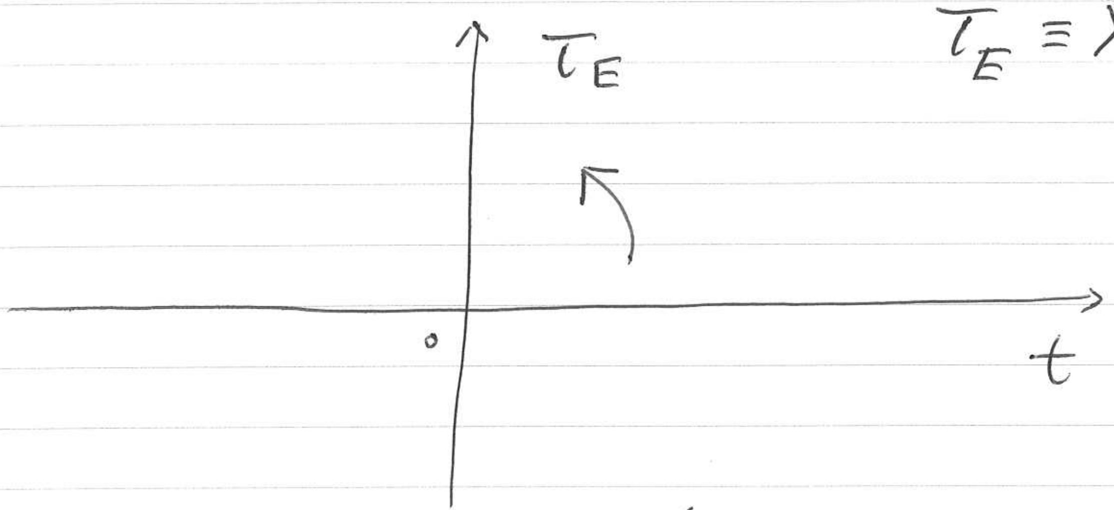
<u>Example</u> <u>Free real scalar field</u>

Euclidean (Wick-rotated) theory

Introduce $\bar{x} \equiv (\bar{x}^0, \bar{\mathbf{x}}) = (i x^0, \bar{\mathbf{x}})$

i.e. $t = -i\tau_E$ (Wick rot.)

$$\tau_E \equiv X^0_E \equiv \bar{X}^0$$




Then: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$

$$\rightarrow \mathcal{L}_E = -\frac{1}{2}\bar{\partial}_\mu\phi\,\bar{\partial}^\mu\phi - \frac{m^2}{2}\phi^2$$

$$\bar{\partial}_\mu\phi\,\bar{\partial}^\mu\phi = \left(\frac{\partial\phi}{\partial X^0_E}\right)^2 + \frac{\partial\phi}{\partial X^i_E}\frac{\partial\phi}{\partial X^i_E}$$

$$W_E[J] = N\int \mathcal{D}\phi\, e^{\int d^4\bar{x}\left(\mathcal{L}_E + J(\bar{x})\phi(\bar{x})\right)}$$

$$\int d^4\bar{x}\,\bar{\partial}_\mu\phi\,\bar{\partial}^\mu\phi = \int d^4\bar{x}\,d^4\bar{y}\,\phi(\bar{y})\,\delta^{(4)}(\bar{x}-\bar{y})\times$$
$$\times\,\bar{\partial}^x_\mu\,\bar{\partial}^\mu_y\,\phi(\bar{x})$$

$$\Rightarrow W_E[J] = N\int\mathcal{D}\phi\, e^{-\frac{1}{2}\int d^4\bar{x}\,d^4\bar{y}\,\phi(\bar{y})\,A(\bar{y},\bar{x})\,\phi(\bar{x}) + \int d^4\bar{x}\,J(\bar{x})\phi(\bar{x})}$$