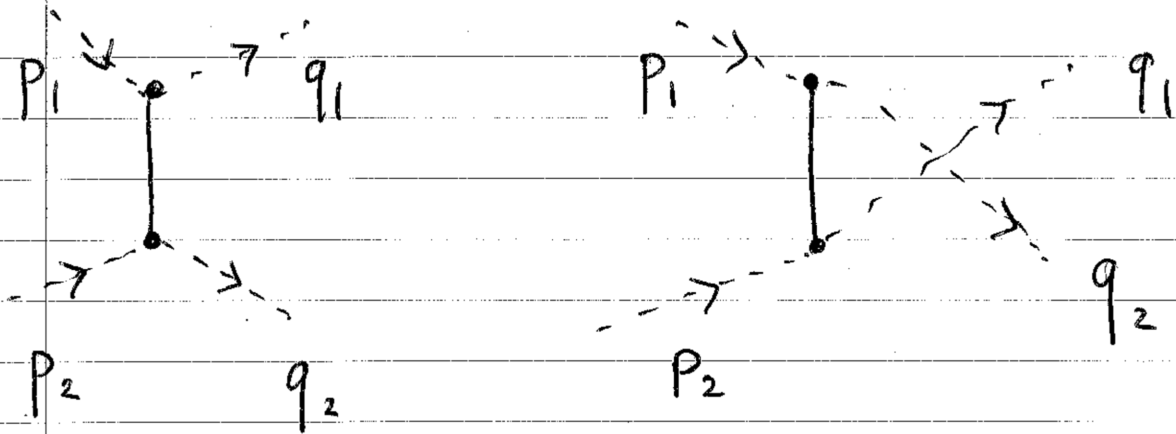
Examples:

- scalar Yukawa theory at $O(g^2)$:



$$\mathcal{M}_{fi} = (-ig)^2 \left[\frac{1}{t - m^2 + i\varepsilon} + \frac{1}{u - m^2 + i\varepsilon} \right]$$

$$t = (p_1 - q_1)^2 \qquad u = (p_1 - q_2)^2$$

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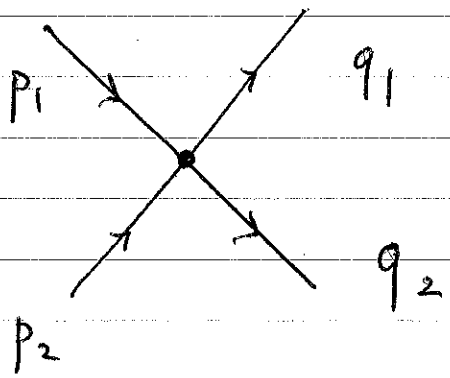
- $\lambda \phi^4$ theory $\left(\mathcal{L}_{int} = -\frac{\lambda}{4!} \phi^4 \right)$.

At $O(\lambda)$:



$$\mathcal{M}_{fi} = -i\lambda + O(\lambda^2)$$

$$d\sigma = \frac{\lambda^2}{64\pi^2 E_{CMF}^2} \, d\Omega \qquad d\Omega = \sin\theta \, d\theta \, d\varphi .$$

Scalar Yukawa theory (continued):

$$t = (p_1 - q_1)^2 \qquad \underline{CMF}: \ \bar{p}_1 + \bar{p}_2 = 0$$

$$\bar{q}_1 + \bar{q}_2 = 0$$

$$u = (p_1 - q_2)^2 \qquad |\bar{p}_1| = |\bar{q}_1|$$

$$p_1 = \left(\sqrt{\bar{p}_1^{\,2} + M^2}, \ \bar{p}_1 \right)$$

$$q_1 = \left(\sqrt{\bar{q}_1^{\,2} + M^2}, \ \bar{q}_1 \right)$$

If $\bar{q} \equiv \bar{p}_1 - \bar{q}_1$, $\bar{q}_* = \bar{p}_1 - \bar{q}_2$,

$$t = -(\bar{p}_1 - \bar{q}_1)^2 \equiv -\bar{q}^{\,2}$$

$$u = -\bar{q}_*^{\,2} \quad \Rightarrow$$

$$\mathcal{M}_{fi} = g^2 \left[\frac{1}{\bar{q}^{\,2} + m^2} + \frac{1}{\bar{q}_*^{\,2} + m^2} \right].$$

In QM, with $H = \dfrac{\bar{p}_1^{\,2}}{2M} + \dfrac{\bar{p}_2^{\,2}}{2M} + V(|\bar{r}_2 - \bar{r}_1|)$,

$$V(r) = g^2 \frac{e^{-mr}}{r} \Rightarrow \mathcal{M}_{Born} \sim \langle f | V | i \rangle$$

$$\mathcal{M}_{Born} \sim \int d^3x\, e^{-i\bar{q}\bar{r}}\, g^2 \frac{e^{-mr}}{r} \sim$$

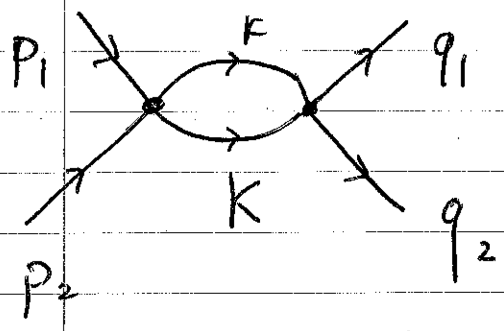
$$\sim g^2 \frac{1}{\bar{q}^2 + m^2}$$ (Do the explicit computation !)

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - M^2 \varphi^* \varphi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - g \varphi^* \varphi \phi$$

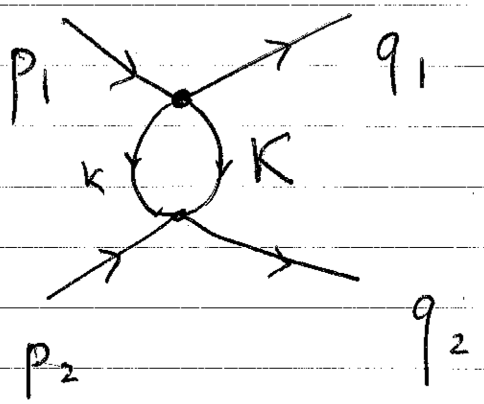
We have interaction via exchange by scalar ϕ of mass m.

<u>Note</u>: the term $\frac{1}{q_*^2 + m^2}$ from the u-channel arises from the exchange interaction in QM (final states are indistinguishable, must use symmetrized $\psi = \frac{1}{\sqrt{2}} \left(\varphi_1(r_1) \varphi_2(r_2) + \varphi_1(r_2) \varphi_1(r_1) \right)$).

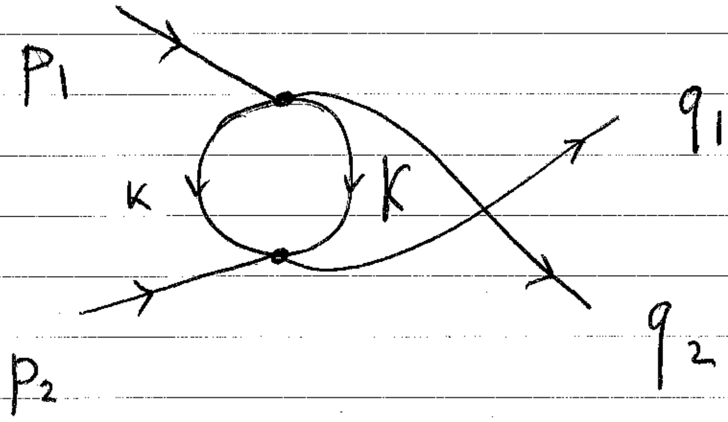
At $O(\lambda^2)$:



$p_1 + p_2 = k + K$



$p_1 - q_1 = k + K$



$p_1 - q_2 = k + K$

Each diagram contributes to $\mathcal{M}_{fi}$:

$$\frac{(-i\lambda)^2}{2!}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{K^2-m^2+i\varepsilon}$$

At $|k| \to \infty$ limit: $\displaystyle\int \frac{d^4k}{k^4} \sim \int \frac{dk}{k} \sim$

$\sim \ln|K| \to \infty$ Disaster.

Introduce cutoff at $|k| = \Lambda$.

Then:

$$\mathcal{M}_{fi} = -i\lambda + iC\lambda^2 \left[\ln\frac{\Lambda^2}{s} + \ln\frac{\Lambda^2}{t} + \ln\frac{\Lambda^2}{u} \right] + O(\lambda^3)$$

(For details of evaluating integrals see A. Zee, QFT in Nutshell, III. 2)

Here C is a constant and we put $m = 0$ for simplicity. Now let:

$$-i\lambda_{phys_0} \equiv -i\lambda + iC\lambda^2 L_0 + \dots \quad (*)$$

$$L_0 \equiv \ln\frac{\Lambda^2}{s_0} + \ln\frac{\Lambda^2}{t_0} + \ln\frac{\Lambda^2}{u_0}.$$

$$-i\lambda_{phys_1} = -i\lambda + iC\lambda^2 L_1 + \dots (**)$$

$$L_1 \equiv \ln\frac{\Lambda^2}{s} + \ln\frac{\Lambda^2}{t} + \ln\frac{\Lambda^2}{u}$$

Solve $(*)$ for λ perturbatively.

$$\lambda = a_1 \lambda_{p_0} + a_2 \lambda_{p_0}^2 + \dots$$

$$-i\lambda_{p_0} = -i a_1 \lambda_{p_0} - i a_2 \lambda_{p_0}^2 + \dots +$$

$$+ iC \lambda_{p_0}^2 a_1^2 L_0 + O(\lambda_{p_0}^3)$$

$$\Rightarrow a_1 = 1 \qquad a_2 = C L_0$$

$$\Rightarrow \lambda = \lambda_{p_0} + C L_0 \lambda_{p_0}^2 + \dots$$

Now subst. into $\lambda_{p_1} = \dots$

$$\lambda_{p_1} = \lambda_{p_0} + C L_0 \lambda_{p_0}^2 - C L_1 \lambda_{p_0}^2 + \dots$$

$$\Rightarrow \boxed{\lambda_{p_1} = \lambda_{p_0} + \lambda_{p_0}^2 C \left[\ln \frac{s_1}{s_0} + \ln \frac{t_1}{t_0} + \ln \frac{u_1}{u_0} \right] + O(\lambda_{p_0}^3)}$$

- <u>Remark</u>: λ_{phys} (or $\mathcal{M}_{fi}^{phys}$) should be independent of Λ $\Rightarrow$

$$\frac{d\,\lambda_{phys}}{d\ln\Lambda} = 0 \qquad \left(d\ln\Lambda = \frac{d\Lambda}{\Lambda} \right)$$

$$\frac{d}{d\ln\Lambda}\left(\mathcal{M}_{fi}\right) = \frac{d}{d\ln\Lambda}\left[-i\lambda + ic\lambda^2\left(3\ln\Lambda^2 - \ln s t u\right)\right] + O(\lambda^3)$$

$$-i\frac{d\lambda}{d\ln\Lambda} + ic\,2\lambda\frac{d\lambda}{d\ln\Lambda}\left(\cdots\right) + ic\lambda^2\cdot 6 = 0$$

$$\sim O(\lambda^3)$$

$$\Rightarrow \boxed{\Lambda\frac{d\lambda}{d\Lambda} = 6c\lambda^2 + O(\lambda^3)}$$

$$\Rightarrow \lambda = \lambda(\Lambda).$$