We now sum over final states:

$$\sum_{n_j} \to \frac{V}{(2\pi)^3} \int d^3 q_j$$

<u>Recall</u>: in finite vol, $q_i = \frac{2\pi}{L} n_i$,

$$\int d^3 q \to \left(\frac{2\pi}{L}\right)^3 \sum_n$$

This gives the measure:

$$d\Pi_n = (2\pi)^4 \delta^{(4)}(P_f - P_i) \prod_{j=1}^{n} \frac{d^3 q_j}{(2\pi)^3 2\omega_{q_j}}$$

Defining the <u>rate</u> $\Gamma_n = P_n / T$, we get

$$\Gamma_n = \frac{1}{2\omega_{p_i}} \int d\Pi_n \, |\mathcal{M}_{fi}|^2$$

- rate of decay into n-particle state.

Total decay width

$$\Gamma = \frac{1}{2\omega_{p_i}} \sum_n \int d\Pi_n \, |\mathcal{M}_{fi}|^2$$

Half-life $\tau = 1/\Gamma$

If the initial particle is at rest,

$\omega_{p_i} = m$ and

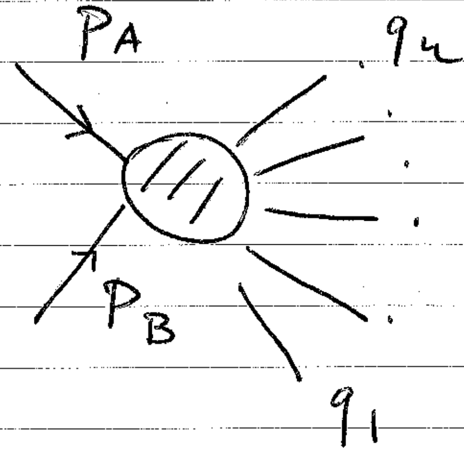
$$\Gamma_{rest} = \frac{1}{2m} \sum_n \int d\Pi_n \, |\mathcal{M}_{fi}|^2$$

$$\Rightarrow \tau = \tau_{rest} \frac{\omega_{p_i}}{m} = \tau_{rest} \frac{\mathcal{E}}{mc^2} = \tau_{rest}\gamma$$

$$\tau = \tau_{rest} / \sqrt{1-\bar{v}^2} > \tau_{rest}.$$

<u>Cross-sections</u>

$|i\rangle = |p_A \, p_B\rangle$



$|f\rangle = |q_1 \cdots q_n\rangle$

Trans. prob. per unit time per flux

$$d\sigma = \frac{1}{F} \frac{d\Pi_n}{4\omega_{\bar{A}} \omega_{\bar{B}} V} |\mathcal{M}_{if}|^2$$

(differential cross-section)

$$\text{Flux } F = \frac{|\bar{v}_{rel}|}{V} = \frac{|\bar{v}_A - \bar{v}_B|}{V} =$$

$$= \frac{|\bar{p}_A/\mathcal{E}_A - \bar{p}_B/\mathcal{E}_B|}{V}$$

Here $p_A = (\mathcal{E}_A, \bar{p}_A)$, $p_B = (\mathcal{E}_B, \bar{p}_B)$

$\mathcal{E}_A \equiv \omega_A$, $\mathcal{E}_B \equiv \omega_A$; $\mathcal{E}^2 = \bar{p}^2 + m^2$.

Can show (see p. 36) that

$$F = \frac{\sqrt{(p_A p_B)^2 - m_A^2 m_B^2}}{\mathcal{E}_A \mathcal{E}_B V}$$

$$\Rightarrow d\sigma = \frac{1}{4\sqrt{(p_A p_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{fi}|^2 d\Pi_n$$

Differential cross-section for $2 \to 2$ scattering.

In CMF : $\bar{p}_A + \bar{p}_B = 0$

$\Rightarrow \bar{q}_1 + \bar{q}_2 = 0$ in CMF

In detail:

$$\left|\frac{\bar{p}_1}{\varepsilon_1}-\frac{\bar{p}_2}{\varepsilon_2}\right| = \sqrt{\frac{\bar{p}_1^{\,2}}{\varepsilon_1^2}+\frac{\bar{p}_2^{\,2}}{\varepsilon_2^2}-\frac{2\bar{p}_1\bar{p}_2}{\varepsilon_1\varepsilon_2}}$$

$$= \sqrt{\frac{\bar{p}_1^{\,2}\varepsilon_2^2+\bar{p}_2^{\,2}\varepsilon_1^2-2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2}{\varepsilon_1^2\varepsilon_2^2}}.$$

But: $(p_1p_2)^2-m_1^2m_2^2 = (\varepsilon_1\varepsilon_2-\bar{p}_1\bar{p}_2)^2-m_1^2m_2^2$

$$= \varepsilon_1^2\varepsilon_2^2-2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2+\bar{p}_1^{\,2}\bar{p}_2^{\,2}-m_1^2m_2^2 =$$

$$= (\bar{p}_1^{\,2}+m_1^2)(\bar{p}_2^{\,2}+m_2^2)+\bar{p}_1^{\,2}\bar{p}_2^{\,2}-m_1^2m_2^2-$$

$$-2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2 =$$

$$= \underbrace{\bar{p}_1^{\,2}\bar{p}_2^{\,2}+m_1^2\bar{p}_2^{\,2}}+\underbrace{\bar{p}_1^{\,2}\bar{p}_2^{\,2}+m_2^2\bar{p}_1^{\,2}}-$$

$$-2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2 =$$

$$= \varepsilon_1^2\bar{p}_2^{\,2}+\varepsilon_2^2\bar{p}_1^{\,2}-2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2$$ as above.

$\{ |\bar{p}_A| = |\bar{p}_B| = |\bar{q}_1| = |\bar{q}_2| \underbrace{\text{elastic scattering}} \}$

For simplicity, assume $m_A = m_B = m_1 = m_2 \equiv m$

Since $p_A + p_B = q_1 + q_2$ and $p_A^2 = m^2$

etc $\Rightarrow p_A p_B = q_1 q_2 = \mathcal{E}_1 \mathcal{E}_2 - \bar{q}_1 \bar{q}_2$.

Now: $(p_A p_B)^2 - m^4 = (\mathcal{E}_1 \mathcal{E}_2 - \bar{q}_1 \bar{q}_2)^2 -$

$- (\mathcal{E}_1^2 - \bar{q}_1^2)(\mathcal{E}_2^2 - \bar{q}_2^2) = \left(\text{since } \bar{q}_1 = -\bar{q}_2 \right) =$

$= (\mathcal{E}_1 \mathcal{E}_2 + \bar{q}_1^2)^2 - (\mathcal{E}_1^2 - \bar{q}_1^2)(\mathcal{E}_2^2 - \bar{q}_1^2) =$

$= (\mathcal{E}_1 + \mathcal{E}_2)^2 \bar{q}_1^2 = \mathcal{E}_{CMF}^2 \bar{q}_1^2$

$$\mathcal{E}_{CMF} = (\mathcal{E}_1 + \mathcal{E}_2)_{CMF}$$

Now consider:

$$\int d\Pi_2 = \int \frac{d^3 q_1}{(2\pi)^3} \frac{1}{2\mathcal{E}_1} \int \frac{d^3 q_2 \, (2\pi)^4 \delta^{(4)}(p_A + p_B - q_1 - q_2)}{2\mathcal{E}_2 (2\pi)^3}$$

1) Integrate over $d^3 q_2$ using $\delta^{(3)}(\bar{p}_A + \bar{p}_B - \bar{q}_1 - \bar{q}_2)$

$\Rightarrow$ this just means $\bar{q}_1 = -\bar{q}_2$

$$\int d\Pi_2 = \int \frac{d^3 q_1}{(2\pi)^3} \frac{1}{2\mathcal{E}_1 \, 2\mathcal{E}_2} 2\pi \delta(\mathcal{E}_{CMF} - \mathcal{E}_1 - \mathcal{E}_2)$$

$$= \int \frac{d\Omega \, d|\vec{q}_1| \, |q_1|^2}{(2\pi)^3 (2\mathcal{E}_1 \, 2\mathcal{E}_2)} 2\pi \delta(\mathcal{E}_{CMF} - \mathcal{E}_1 - \mathcal{E}_2)$$

Here $\mathcal{E}_1 = \mathcal{E}_2 = \sqrt{\vec{q}_1^{\,2} + m^2}$.

<u>Recall</u>: $\delta(F(x)) = \sum_i \frac{\delta(x - x_i^0)}{|F'(x_i^0)|}$,

where x_i^0 are zeros of F: $F(x_i^0) = 0$.

$$\Rightarrow \int d\Pi_2 = \frac{1}{16\pi^2} \int d\Omega \frac{\vec{q}_1^{\,2} d|q_1|}{\mathcal{E}_1^2} \delta\left(\mathcal{E}_{CMF} - 2\sqrt{\vec{q}_1^{\,2} + m^2}\right)$$

$$= \frac{1}{16\pi^2} \int d\Omega \frac{|q_1|}{2\mathcal{E}_1} = \frac{1}{16\pi^2} \int d\Omega \frac{|q_1|}{\mathcal{E}_{CMF}}.$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|\mathcal{M}_{fi}|^2}{\mathcal{E}_{CMF}^2} = \frac{1}{64\pi^2} \frac{|\mathcal{M}_{fi}(s,t,u)|^2}{s}.}$$