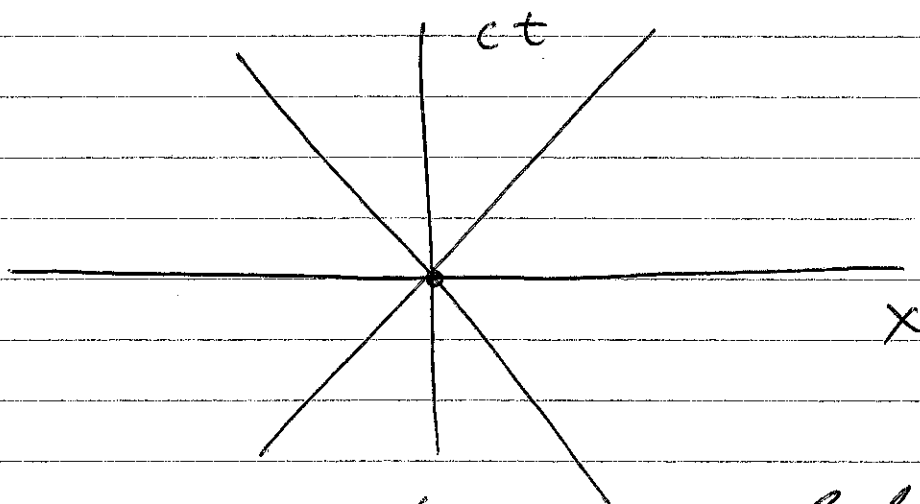


①

Causality

In SR, we have the light-cone



We can compute the probability for a particle in QM to travel from $\bar{x}=0$, $t=0$ to (\bar{x}, t) outside the light-cone.

$$P = |A|^2,$$

$$A = \langle x | e^{-i\hat{H}t} | 0 \rangle =$$

$$= \int d^3p \langle x | e^{-i\hat{H}t} | p \rangle \langle p | 0 \rangle$$

$$\text{Recall: } \langle x | p \rangle = \frac{1}{(2\pi)^{3/2}} e^{ip\bar{x}}$$

$$A = \int d^3p \frac{e^{ip\bar{x}}}{(2\pi)^3} e^{-i\varepsilon_p t}$$

Here:

$$\hat{H}|p\rangle = \varepsilon_p |p\rangle$$

$$\varepsilon_p = \sqrt{p^2 + m^2}$$

$$A = \frac{1}{2\pi^2 |\bar{x}|} \int_0^{\infty} dp p \sin p |\bar{x}| e^{-i\epsilon p t} \quad (2)$$

$$z = \sqrt{p^2 + m^2} \quad dz = p dp / z$$

$$\Rightarrow A = \frac{1}{2\pi^2 |\bar{x}|} \int_m^{\infty} dz \cdot z e^{-itz} \sin \left[|\bar{x}| \sqrt{z^2 - m^2} \right]$$

PBM-I - 2.5.42.3

$$A = \frac{i m^2 t}{2\pi^2} \frac{K_2 \left(m \sqrt{\bar{x}^2 - t^2} \right)}{\bar{x}^2 - t^2}$$

clearly non-zero for $t^2 - \bar{x}^2 < 0 \dots$

Deep space-like region $|\bar{x}| \gg t$

$A \sim e^{-m|\bar{x}|}$: small but finite.

This is problematic.

More generally, in QM, observables are Hermitian operators not attached to space-time points \Rightarrow this results in "action at a distance".

③

Observers separated by space-like intervals and measuring $\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2$ with $[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] \neq 0$ violate causality.

To avoid "action at a distance", promote $\hat{\mathcal{O}}$ to a field $\hat{\mathcal{O}}(x)$ - operator-valued (generalized) functions of space-time points x . Then causality is preserved if we demand

$$[\hat{\mathcal{O}}_1(x_1), \hat{\mathcal{O}}_2(x_2)] = 0$$

for space-like intervals, $(x_2 - x_1)^2 < 0$.

Electromagnetic field as Hamiltonian system

$$\text{Maxwell eqs} \quad \left\{ \begin{array}{l} \partial_\mu F^{\mu\nu} = \mu_0 J^\nu \\ \partial_\mu F_{\nu\lambda} = 0 \quad (\text{Bianchi}) \\ \text{constraints} \rightarrow \end{array} \right.$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad A_\mu \rightarrow A_\mu - \partial_\mu f$$

gauge invar.

(4)

Choose convenient gauge

e.g. $A^0 = 0$, $\text{div } \vec{A} = 0 \Rightarrow$ eom are simple: $\square \vec{A} = 0$

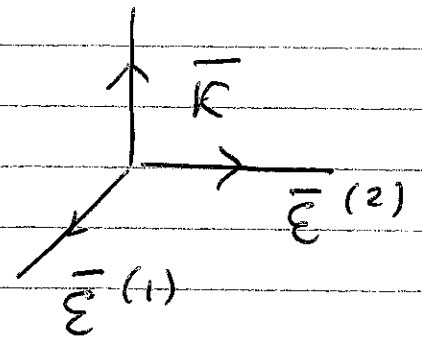
Look for solutions in the form

$$A_i(t, \vec{x}) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3} N_k \left(e^{i\vec{k}\vec{x}} \epsilon_i^{(\lambda)}(\vec{k}) A(k,t) + \text{c.c.} \right)$$

↑
normalization

Since $\partial_i A_i = 0 \Rightarrow \vec{k} \cdot \vec{\epsilon}^{(\lambda)} = 0$: can

choose $\vec{\epsilon}^{(\lambda)} \vec{\epsilon}^{(\lambda')} = \delta^{\lambda\lambda'}$



Then $\square A_i = 0$ gives

$$\ddot{A}^{(\lambda)}(k,t) + |\vec{k}|c^2 A^{(\lambda)}(k,t) = 0$$

$$\Rightarrow A_i(t, \vec{x}) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3} \epsilon_i^{(\lambda)} \left[a_\lambda(k) e^{-i\vec{k}\vec{x}} + a_\lambda^* e^{i\vec{k}\vec{x}} \right]$$

Here $kx = k^0 x^0 - \vec{k}\vec{x} = \omega_k t - \vec{k}\vec{x}$, $\omega_k = |\vec{k}|c$

5

The energy of the field

$$\mathcal{E} = \frac{1}{2} \int d^3x (\vec{E}^2 + \vec{B}^2) \text{ is}$$

$$\mathcal{E} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} \omega_k a_{\lambda}^*(k) a_{\lambda}(k)$$

with $N_k = c / \sqrt{2\omega_k}$.

Introducing

$$\left. \begin{aligned} Q_{\lambda,k} &= \frac{a_{\lambda} + a_{\lambda}^*}{\sqrt{2\omega_k}} \\ P_{\lambda,k} &= -\frac{i\omega_k (a_{\lambda} - a_{\lambda}^*)}{\sqrt{2\omega_k}} \end{aligned} \right\}$$

we find

$$\mathcal{E} = \sum_{\lambda} \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} (P_{\lambda,k}^2 + \omega_k^2 Q_{\lambda,k}^2)$$

Major discovery: free electromagnetic field is an infinite collection of harmonic oscillators.

Canonical quantization: $[\hat{P}_{\lambda,k}, \hat{Q}_{\lambda',k'}] = -i\hbar \delta_{\lambda\lambda'} \delta_{kk'}$
 $[\hat{P}_{\lambda',k'}, \hat{P}_{\lambda,k}] = 0, [\hat{Q}_{\lambda,k}, \hat{Q}_{\lambda',k'}] = 0$