KG eq. also has problems with interpreting $|\psi|^2$ as prob. density (as a conseq. of containing second time derivative). In 1928, Dirac found rel. wave eq. with one time deriv.

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 .$$

For the hydrogen atom, Dirac eq. gives (Charles G. Darwin, 1928; W. Gordon, 1928)

$$E_{nj} = mc^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{j+1/2} - \frac{3}{4}\right) + \cdots\right]$$

where $j = 1/2, 3/2, \ldots n-1/2$. $(\vec{J} = \vec{L} + \vec{S})$

Dirac eq. predicts fine structure term correctly (!)

Moreover, Dirac eq has the correct value of the electron's magnetic moment built in (assuming min coupling).

$$(\mu_s)_z = -g_s \mu_B m_s$$

$m_s = \pm 1/2$, $\mu_B = \dfrac{e\hbar}{2m}$ (Bohr magneton)

g_s : gyromagnetic ratio

Naively: rotating uniformly charged ball of radius a and charge q

$$\left.\begin{array}{l} \mu = \frac{1}{5} q \omega a^2 \\ L = I\omega = \frac{2}{5} m a^2 \omega \end{array}\right\} \mu = \frac{q}{2m} L$$

With $L \to \pm \hbar/2 \Rightarrow \mu = \pm \dfrac{1}{2} \dfrac{e\hbar}{2m} =$

$$= \pm \frac{1}{2} \mu_B$$

But experiments give $\mu_e = \mu_B$, i.e. $g_s = 2$. Dirac eq. gives $g_s = 2$.

<u>Note</u>: one can add a term to Dirac eq. to modify g_s. Such term would be non-renormalizable in QED, however.

Dirac eq. also predicted antiparticles (positrons) but still had problems if interpreted as a single-particle equation. This is not an accident.

Observation 1

Consider a scattering process e.g.

$$p + p \to p + p$$

Since in SR $E = m_* c^2$, we may expect for high enough E that new particles of mass m_* will appear in the final state. Indeed, for $E > m_\pi c^2$ ($m_\pi c^2 \sim 140$ MeV):

$$p + p \to p + p + \pi^0$$

is observed.

For $E > 2 m_p c^2$ (~ 2 GeV)

$$p + p \to p + p + p + \bar{p}$$

and so on.

<u>Observation 2</u> : „zooming in" at small distances – Heisenberg uncertainty principle tells us $\Delta x \gtrsim \hbar/\Delta p$, so localizing an object within $\Delta x \sim L$ we have $\Delta p \gtrsim \hbar/L$ or, in terms of energy ($E = \sqrt{p^2c^2+m^2c^4} \simeq pc$ for high p) : $\Delta E \gtrsim \hbar c/L$

$$\Rightarrow \quad \frac{\Delta E}{mc^2} \gtrsim \frac{\hbar}{mcL} = \frac{\lambdabar}{L}$$

Trying to „resolve" distances

$$L \lesssim \lambdabar$$

will require $\Delta E \gtrsim mc^2$

$\Rightarrow$ particle production.

- In atom, $\lambdabar_e \sim 10^{-10}\,\text{cm} \sim 10^{-2}\,a_B$

$\Rightarrow$ electron well localized, $\lambdabar < L$

$\Rightarrow$ rel. corrections small

- Light quarks in proton

$m_u \sim m_d \sim 10$ MeV $\quad \lambdabar \sim 20\,\text{fm} > L_{prot} \sim 1\,\text{fm}$.

Observation 3 : even for low-energy processes, one is forced to consider all states. In pert. theory, if

$$H = H_0 + \delta V \Rightarrow E \to E_0 + \delta E_0 ,$$

$$\delta E_0 = \langle 0 | \delta V | 0 \rangle + \sum_n \frac{|\langle 0 | \delta V | n \rangle|^2}{E_0 - E_n} + \dots$$

for e.g. ground state energy.
Intermediate states will contribute

$$\frac{\text{typical energies in problem}}{\text{typical energy denominator}} \sim \frac{mv^2}{mc^2}$$

$$\sim v^2/c^2$$

$\Rightarrow$ the same order as rel. kinematics
In general, intermediate states of all energies give comparable contrib. to rel. kinematics. (Not in hydrogen atom $\Rightarrow$ success of Dirac eq.)
Moral: rel. quantum theory is necessarily a rel. many-body problem.

Hydrogen spectroscopy (continued)

Experiments of late 1940s detected additional structures in the hydrogen atom spectrum (Lamb - Retherford, 1947) as well as the anomalous magnetic moment of the electron (Kusch, 1947). They were successfully explained by, resp., Bethe and Schwinger (1947-48) using QED.

$$\mu_e = \mu_B \left(1 + \frac{\alpha}{2\pi} + O(\alpha^2) \right)$$

As of 2018, for $a = \frac{\mu_e}{\mu_B} - 1 = \frac{g-2}{2}$ we have

$$a_{theory} = (11\,596\,521\,818 \pm 9) \cdot 10^{-13}$$

$$a_{exp} = (11\,596\,521\,809 \pm 3) \cdot 10^{-13}$$