Wick's theorem

__Recall__: for $\hat{H} = \hat{H}_0 + \hat{H}_I$, we introduced interact. picture (or inter. represent.), where operators $\hat{O}_{i.p.}(x)$ evolve in time with $\hat{H}_0$:

$$\frac{d}{dt}\hat{O}_{ip}(x) = \frac{i}{\hbar}\left[\hat{H}_0, \hat{O}_{ip}(x)\right]$$

– as in free theory, but the states obey

$$i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle_{ip} = \hat{H}_I^{ip}\,|\varphi(t)\rangle_{ip}.$$

This eq. can be converted into integral eq. and solved by iterations:

$$|\varphi(t_f)\rangle_{ip} = T\exp\left[\frac{-i}{\hbar}\int_{t_i}^{t_f}\hat{H}_I^{ip}(\bar{t})\,d\bar{t}\right]|\varphi(t_i)\rangle_{ip}.$$

(Dyson, 1949).

__Scattering problem__:

$t_i \to -\infty \qquad |\varphi(t_i)\rangle_{ip} \to |i\rangle$

$t_f \to +\infty \qquad |\varphi(t_f)\rangle_{ip} \to |f\rangle$

$$|f\rangle = \hat{S}|i\rangle$$

$$\hat{S} = T\exp\left[-\frac{i}{\hbar}\int d^4x\, \hat{\mathcal{H}}_I^{i.p.}(x)\right]$$

$$S_{fi} = \langle f|\hat{S}|i\rangle \qquad \hat{S}^\dagger\hat{S} = \mathbb{1} :$$

Unitary operator (conserv. of prob.).

Convenient to define $\hat{T}$: $\hat{S} = 1 + i\hat{T}$

<u>Exercise</u>: show that unitarity implies

$$2\,\mathrm{Im}\,T_{ii} = \sum_f |T_{if}|^2$$

- Computing S_{fi} is equiv. to computing

$$\langle f|\,T\left(\mathcal{H}_I^{ip}(x_1)\cdots\mathcal{H}_I^{ip}(x_n)\right)|i\rangle$$

e.g. for $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$,

$$\mathcal{H}_I(x) = \frac{\lambda}{4!}\phi^4(x)$$

$$\mathcal{H}_I^{i.p.}(x) = \frac{\lambda}{4!}\phi_{ip}^4(x) \quad : \text{expand in } a, a^\dagger.$$

- Consider $\langle 0|\,T\phi_{ip}(x)\,\phi_{ip}(y)\,|0\rangle$

$$\phi_{ip}(x) = \phi^+_{ip} + \phi^-_{ip},$$

$$\phi^+_{ip} = \int d^3\tilde{p}\, a_{\bar{p}}\, e^{-ipx} \qquad \phi^-_{ip} = \int d^3\tilde{p}\, a^+_{\bar{p}}\, e^{ipx}$$

By construction, $\phi^+_{ip}|0\rangle = 0$ and

$\langle 0|\phi^-_{ip} = 0.$

- $x^0 > y^0$: $T\phi_{ip}(x)\phi_{ip}(y) = \phi(x)\phi(y) =$

$$= \left(\phi^+_{ip}(x) + \phi^-_{ip}(x)\right)\left(\phi^+_{ip}(y) + \phi^-_{ip}(y)\right) =$$

$$= \phi^+_{ip}(x)\phi^+_{ip}(y) + \phi^-_{ip}(x)\phi^+_{ip}(y) + \phi^-_{ip}(x)\phi^-_{ip}(y)$$

$$+ \phi^-_{ip}(y)\phi^+_{ip}(x) + [\phi^+_{ip}(x), \phi^-_{ip}(y)] =$$

$$= \underbrace{:\phi_{ip}(x)\phi_{ip}(y):}_{\text{all terms are } \sim a^+a} + [\phi^+_{ip}(x), \phi^-_{ip}(y)]$$

- $x^0 < y^0$: $T\phi_{ip}(x)\phi_{ip}(y) = \phi_{ip}(y)\phi_{ip}(x) =$

$$= :\phi(x)\phi(y): + [\phi^+_{ip}(y), \phi^-_{ip}(x)].$$

Define the contraction of $\phi(x), \phi(y)$:

$$\wick{\c\phi_{ip}(x)\c\phi_{ip}(y)} = \begin{cases} [\phi^+_{ip}(x)\,\phi^-_{ip}(y)], & x^0 > y^0 \\ [\phi^+_{ip}(y)\,\phi^-_{ip}(x)], & x^0 < y^0 \end{cases}$$

Observe: $\wick{\c\phi_{ip}(x)\c\phi_{ip}(y)} = D_F(x-y)$.

Then:

$$T\big(\phi_{ip}(x)\,\phi_{ip}(y)\big) = :\phi_{ip}(x)\phi_{ip}(y): + \wick{\c\phi_{ip}(x)\c\phi_{ip}(y)}.$$

In general:

$$T\big(\phi_{ip}(x_1)\cdots\phi_{ip}(x_n)\big) = :\phi_{ip}(x_1)\cdots\phi_{ip}(x_n): +$$

$$+ :\text{all possible contractions}:$$

(Wick, 1950; Dyson, 1951)

example: $T\big(\phi_{ip}(x_1)\,\phi_{ip}(x_2)\,\phi_{ip}(x_3)\big) =$

$$= :\phi_{ip}(x_1)\phi_{ip}(x_2)\phi_{ip}(x_3): + :\wick{\c\phi_{ip}(x_1)\c\phi_{ip}(x_2)}\phi_{ip}(x_3):$$

$$+ :\phi_{ip}(x_1)\wick{\c\phi_{ip}(x_2)\c\phi_{ip}(x_3)}: + :\wick{\c\phi_{ip}(x_1)\phi_{ip}(x_2)\c\phi_{ip}(x_3)}:$$

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<u>example:</u> $T\left(\phi_{ip}(x_1)\,\phi_{ip}(x_2)\,\phi_{ip}(x_3)\,\phi_{ip}(x_4)\right) =$

$$= :\phi_{ip}(x_1)\,\phi_{ip}(x_2)\,\phi_{ip}(x_3)\,\phi_{ip}(x_4): \; +$$

$$+ :\wick{\c1\phi_{ip}(x_1)\,\c1\phi_{ip}(x_2)}\,\phi_{ip}(x_3)\,\phi_{ip}(x_4): \; +$$

$$+ :\wick{\c1\phi(x_1)\,\phi(x_2)\,\c1\phi(x_3)}\,\phi(x_4): \; +$$

$$+ :\wick{\c1\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\c1\phi(x_4)}: \; +$$

$$+ :\phi(x_1)\,\wick{\c1\phi(x_2)\,\c1\phi(x_3)}\,\phi(x_4): \; +$$

$$+ :\phi(x_1)\,\wick{\c1\phi(x_2)\,\phi(x_3)\,\c1\phi(x_4)}: \; +$$

$$+ :\phi(x_1)\,\phi(x_2)\,\wick{\c1\phi(x_3)\,\c1\phi(x_4)}: \; +$$

$$+ :\wick{\c1\phi(x_1)\,\c1\phi(x_2)\,\c2\phi(x_3)\,\c2\phi(x_4)}: \; +$$

$$+ :\wick{\c1\phi(x_1)\,\c2\phi(x_2)\,\c1\phi(x_3)\,\c2\phi(x_4)}: \; +$$

$$+ :\wick{\c1\phi(x_1)\,\c2\phi(x_2)\,\c2\phi(x_3)\,\c1\phi(x_4)}:$$

Note: $\langle 0|$ any uncontracted normal-ordered $\phi_{ip}(x)\,|0\rangle = 0$

$$\Rightarrow \langle 0|T\left(\phi_{ip}(x_1)\,\phi_{ip}(x_2)\right)|0\rangle = D_F(x_1-x_2)$$

$$\langle 0|T\left(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\right)|0\rangle = 0$$

(the same for $\forall$ number of $\phi_{ip}(x)$ which is odd)

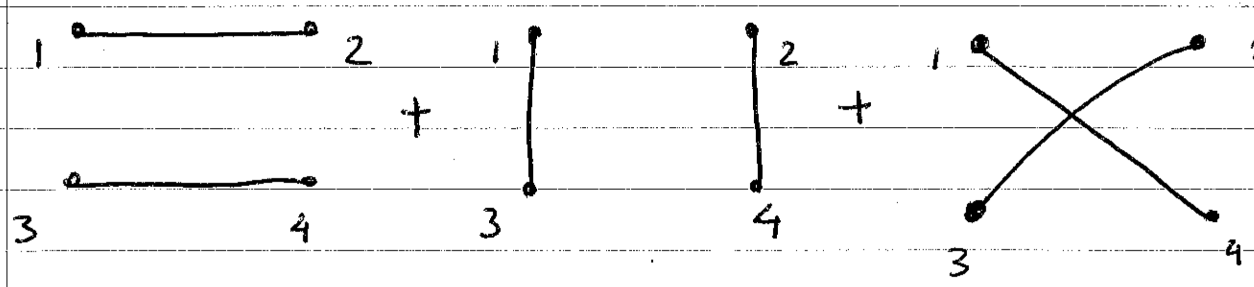
$$\langle 0|T\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right)|0\rangle =$$

$$= D_F(x_1-x_2)\,D_F(x_3-x_4) +$$

$$+ D_F(x_1-x_3)\,D_F(x_2-x_4) +$$

$$+ D_F(x_1-x_4)\,D_F(x_2-x_3) \quad =$$

=



Feynman diagrams.