Interacting Quantum Fields

* Self-interaction, e.g.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$(\Box + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \quad \text{Non-linear eom}$$

* Interaction among different fields

$$\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - M^2\varphi^*\varphi -$$

$$- \frac{1}{2}m^2\phi^2 - \underbrace{g\varphi^*\varphi\phi}$$

Naive perturbative approach:

$$\mathcal{L} = \mathcal{L}_0 + \lambda\mathcal{L}_1$$

$\mathcal{L}_0$: free theory

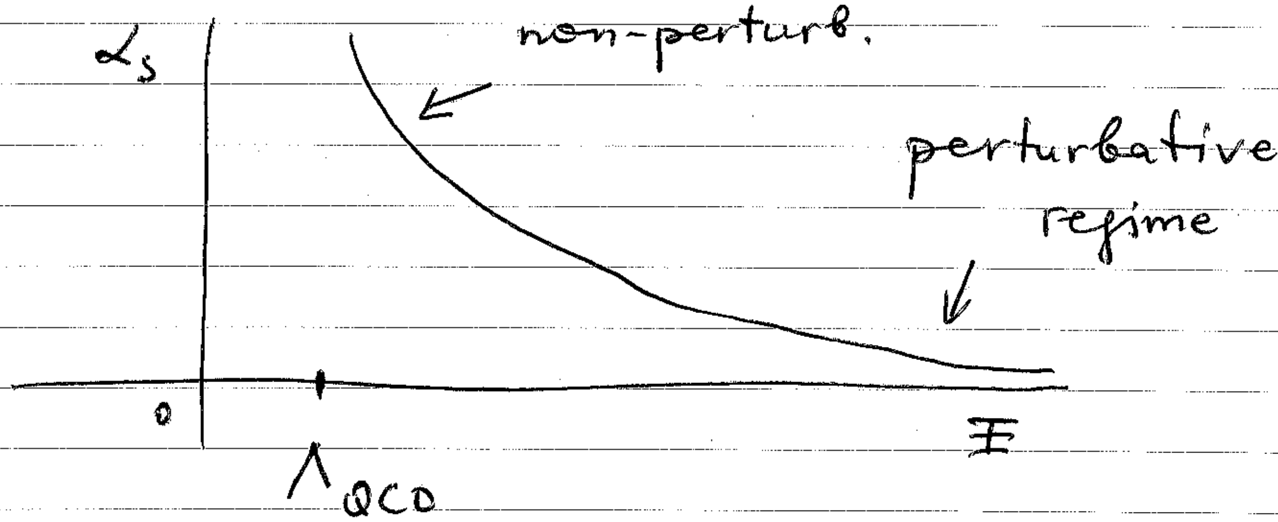
$\mathcal{L}_1$: perturbation „small" w.r.t. $\mathcal{L}_1$.

Criteria of „smallness"

- $\lambda \ll 1$ (?) But λ can be dimensionful..
- $\lambda \ll 1$, λ dimensionless (as in $\lambda\phi^4$ above)

Still problematic, since $\lambda = \lambda(E)$

E.g. in QCD $\alpha_s = g^2/4\pi$



* Non-perturbative methods in QFT are important (e.g. Lattice QCD); they are usually not universal, e.g. LQCD cannot adequately deal with time-dep. problems requiring Mink. space-time (only works with time-indep. situations)
* Perturbative approach gave many important physical predictions.

Interaction picture

Recall:

- Schrödinger picture
  - states are time-dep. $|\varphi(\bar{x},t)\rangle_S$
  - operators $\hat{O}_S$ are time-indep.

$$i\hbar \frac{\partial}{\partial t} |\varphi\rangle_S = \hat{H} |\varphi\rangle_S$$

$$|\varphi(\bar{x},t)\rangle_S = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)} |\varphi(\bar{x},t_0)\rangle_S \equiv$$

$$\equiv \hat{U}(t,t_0) |\varphi(\bar{x},t_0)\rangle_S$$

<u>Physical quantities:</u>

$${}_S\langle \varphi(\bar{x},t)| \hat{O}_S |\varphi(\bar{x},t)\rangle_S = \mathcal{O}(x,t)$$

- Heisenberg picture
  - states are time-indep. $|\varphi\rangle_H$
  - operators $\hat{O}_H$ are time-dep.

$$\langle \varphi(\bar{x},t)| \hat{O}_S |\varphi(\bar{x},t)\rangle_S = \langle \varphi(t_0)| \hat{O}_H |\varphi(t_0)\rangle,$$

where $\hat{O}_H = \hat{U}^+(t)\, \hat{O}_S\, \hat{U}(t)$.

e.o.m. $\frac{d}{dt}\hat{\mathcal{O}}_H = \frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}_H]$

For $\hat{H} = \hat{H}_0 + \hat{H}_I$, introduce

$$\hat{\mathcal{O}}_{int.p.} = \hat{U}_0^+ \hat{\mathcal{O}}_S \hat{U}_0, \quad U_0 = e^{-\frac{i}{\hbar}\hat{H}_0(t-t_0)}$$

E.o.m. for $\hat{\mathcal{O}}_{int.p.}$ – ?

$$\dot{\mathcal{O}}_{int\,p} = \dot{U}_0^+ \mathcal{O}_S U_0 + U_0^+ \mathcal{O}_S \dot{U}_0 =$$

$$= \frac{i}{\hbar} H_0 \mathcal{O}_{int.p.} - \frac{i}{\hbar}\mathcal{O}_{int.p.} H_0 =$$

$$= \frac{i}{\hbar}[H_0, \mathcal{O}_{int.p.}]$$

Moral: $\hat{\mathcal{O}}_{int.p.}(t)$ e.o.m. is the same as in the Heis. picture for free fields

The states are not time-indep.:

$$|\varphi(t)\rangle_S = U_0(t)\,|\varphi(t)\rangle_{int.p.}$$

Since $i\hbar \frac{\partial |\varphi(t)\rangle_s}{\partial t} = (H_0 + H_I)|\varphi(t)\rangle_s$,

we find (check this!):

$$\boxed{\begin{array}{l} i\hbar \frac{\partial}{\partial t} |\varphi(t)\rangle_{int.p.} = \hat{H}_I^{int.p.} |\varphi(t)\rangle_{int.p.}, \\ \text{where } H_I^{int.p.} \equiv U_0^{+} H_I U_0 . \end{array}}$$

Since field operators $\hat{\phi}_{int.p}$ obey free e.o.m., their mode expansion and commut. rel. are the same as in free theory, e.g. for complex scalar field:

$$[\hat{\phi}_{int.p.}(x), \hat{\phi}^{+}_{int.p}(y)] = i\,\Delta(x-y)$$

Need to consider time evolution of the states $|\varphi(t)\rangle_{int.p.}$

Start at $t \to -\infty$ with some initial state $|i\rangle_{int.p.}$ It evolves to $|\varphi(+\infty)\rangle_{int.p.}$

$$|\varphi(+\infty)\rangle_{int.p.} = S|i\rangle$$

If $|\varphi(+\infty)\rangle_{int.p} = \sum_f \alpha_f |f\rangle$

for a complete set of final states, then

$$|\varphi(+\infty)\rangle_{int.p.} = \sum_f S_{fi} |f\rangle, \text{ where}$$

$$S_{fi} = \langle f | \varphi_{int.p.}(+\infty)\rangle = \langle f|S|i\rangle$$

Conservation of probability $\Rightarrow \sum_f |S_{fi}|^2 = 1$

<u>Need to solve evol. eq. for $|\varphi(t)\rangle_{int.p.}$</u>
<u>with initial condition $|\varphi(-\infty)\rangle_{int.p} = |i\rangle$</u>

Write the eq $i\hbar \frac{\partial}{\partial t}|\varphi(t)\rangle_{int.p} = \hat{H}_I^{int.p}|\varphi\rangle_{int.p.}$

as an integral eq:

$$|\varphi(t)\rangle_{in.p} = |i\rangle - \frac{i}{\hbar}\int_{-\infty}^{t} d\tau\, H_I^{in.p.}(\tau)|\varphi(\tau)\rangle_{in.p.}$$

Perturbative solution

$$|\varphi(t)\rangle_{int.p.} = \varphi_0 + \varphi_1 + \cdots$$

(Assuming a small param. in H_I such as $\lambda \ll 1$ or $\alpha_{em} = e^2/4\pi \sim 1/137$)

$$\varphi_0 = |i\rangle$$

$$\varphi_1 = -\frac{i}{\hbar} \int_{-\infty}^{t} d\tau_1 \, H_I^{int.p.}(\tau_1) \, |i\rangle$$

$$\varphi_2 = \left(-\frac{i}{\hbar}\right)^2 \int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \, H_I^{int.p.}(\tau_1) \, H_I^{int.p.}(\tau_2) \, |i\rangle$$

In the limit $t \to +\infty$;

$$S = \sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_n \, H_I^{i.p.}(\tau_1) \cdots H_I^{i.p.}(\tau_n)$$

$$= \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \cdots \int_{-\infty}^{\infty} d\tau_n \, T \, H_I^{ip}(\tau_1) \cdots H_I^{ip}(\tau_n)$$

$$\equiv T\exp\left(-\frac{i}{\hbar} \int_{-\infty}^{\infty} d\tau \, H_I^{ip}(\tau)\right).$$

See e.g. Peskin-Schroder, Sect. 4.2, Fig. 4.1.

For Hamilt. density $\mathcal{H}$ we have

$$S = T \exp\left(-\frac{i}{\hbar}\int d^4x\, \mathcal{H}_I^{i.p.}(x)\right).$$

Need tools to deal with products

$$\int d^4x_1 \cdots \int d^4x_n\, T\, \mathcal{H}_I^{i.p.}(x_1) \cdots \mathcal{H}_I^{i.p.}(x_n)$$

<u>Wick's theorem</u>

Recall: $\hat{\phi}_{i.p.} = \hat{\phi}^{(+)}_{i.p.} + \hat{\phi}^{(-)}_{i.p.}$,

$$\hat{\phi}^{(+)}_{i.p.} = \int d^3\tilde{p}\ \hat{a}_{\bar{p}}\, e^{-ipx}$$

$$\hat{\phi}^{(-)}_{i.p.} = \int d^3\tilde{p}\ \hat{a}^{+}_{\bar{p}}\, e^{ipx}$$

Consider $T(\phi_A(x_1)\,\phi_B(x_2))$ We omit int. p. index

1) $t_1 > t_2$: $T\phi_A(x_1)\phi_B(x_2) = \phi_A(x_1)\phi_B(x_2)$

$= \phi_A^{(+)}(x_1)\,\phi_B^{(+)}(x_2) + \phi_A^{(-)}(x_1)\,\phi_B^{(-)}(x_2) +$