Path Integrals in QFT

In QFT, we are interested in N-point functions

$$G^{(N)}(x_1 \cdots x_N) = \langle 0 | T \hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_N) | 0 \rangle$$

Recall that in QM:

$$\langle x_f | e^{-i\hat{H}(t_f - t_i)/\hbar} | x_i \rangle =$$

$$= N \int_{\substack{x(t_i) = x_i \\ x(t_f) = x_f}} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \mathcal{L}(x, \dot{x})\, dt}$$

Here $|x_i\rangle$ and $|x_f\rangle$ are eigenstates of the position operator. The amplitude is the position space repres. of the Schröd. evolution operator.

The analog of $G^{(2)}(x_1, x_2)$ in QM is

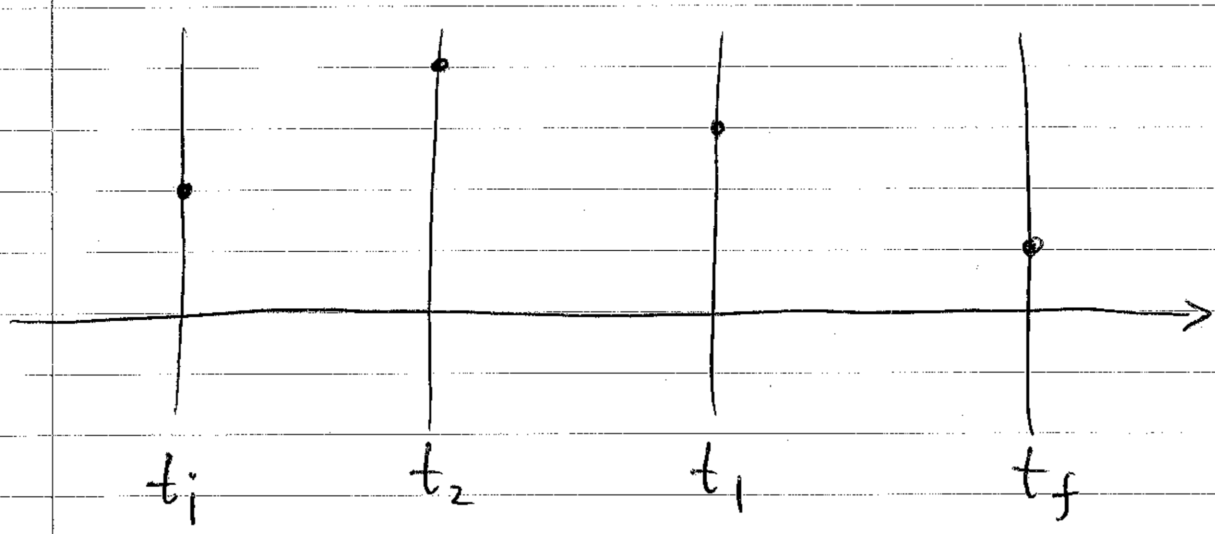
$$\langle x_f\, t_f | T \hat{x}(t_1) \hat{x}(t_2) | x_i\, t_i \rangle =$$

$$= \langle x_f | e^{-i\hat{H}(t_f - t_1)} \hat{X}_s \, e^{-i\hat{H}(t_1 - t_2)} \hat{X}_s \, e^{-i\hat{H}(t_2 - t_i)} | x_i \rangle$$

$$= \int dx_1 \, dx_2 \, \langle x_f | e^{-i\hat{H}(t_f - t_1)} | x_1 \rangle \langle x_1 | \hat{X}_s \, e^{-i\hat{H}(t_1 - t_2)} | x_2 \rangle$$

$$\times \langle x_2 | \hat{X}_s \, e^{-i\hat{H}(t_2 - t_i)} | x_i \rangle$$

Here $t_1 > t_2$



$x(t_2) = x_2 \qquad x(t_1) = x_1$

E.g. $\langle x_f | e^{-iH(t_f - t_1)} | x_1 \rangle =$

$$= N \int \mathcal{D}x(t) \, e^{i \int_{t_1}^{t_f} \mathcal{L} \, dt}$$

$x(t_1) = x_1$
$x(t_f) = x_f$

Integration over x_1, x_2 removes constraints $x(t_1) = x_1$, $x(t_2) = x_2$

$$\Rightarrow \langle x_f t_f | T \hat{x}(t_1) \hat{x}(t_2) | x_i t_i \rangle =$$

$$= \mathcal{N} \int_{\substack{x(t_i)=x_i \\ x(t_f)=x_f}} \mathcal{D}x(t) \; x(t_1) x(t_2) \, e^{i\int_{t_i}^{t_f} \mathcal{L}\, dt}$$

(Same for $t_1 < t_2$.)

We also need to relate $|x_{i,f} \, t_{i,f}\rangle$ to $|0\rangle$ at $t \to \mp\infty$. This gives an overall phase (see Peskin - Schröder, Sect. 4.2)

$$\Rightarrow \langle 0 | T \hat{x}(t_1) \cdots \hat{x}(t_N) | 0 \rangle =$$

$$= \frac{\int \mathcal{D}x(t)\, e^{iS[x]} x(t_1) x(t_2) \cdots x(t_N)}{\int \mathcal{D}x(t)\, e^{iS[x]}}$$

In QFT:

$$\langle 0 | T \hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_N) | 0 \rangle =$$

$$= \mathcal{N} \int \mathcal{D}\phi \; \phi(x_1) \cdots \phi(x_N) \, e^{iS[\phi]}$$

Introduce the generating functional

$$W[J] = \mathcal{N}\int \mathcal{D}\phi \, e^{i\int d^4x\left[\mathcal{L} + J(x)\phi(x)\right]}$$

Then

$$G^{(N)}(x_1 \cdots x_N) = \frac{1}{i^N}\,\frac{\delta^N W[J]}{\delta J(x_1)\cdots\delta J(x_N)}\Bigg|_{J=0}$$

$$\left(W[J]\Big|_{J=0} = W[0] = \langle 0|0\rangle = 1 \quad \Rightarrow \quad \mathcal{N} = \frac{1}{\int \mathcal{D}\phi \, e^{iS}} \right)$$

It is convenient to use

$$Z[J] = -i \ln W[J]$$

$$\Rightarrow \quad G^{(N)}_{\text{connected}}(x_1 \cdots x_N) = \frac{1}{i^{N-1}}\,\frac{\delta^N Z[J]}{\delta J(x_1)\cdots\delta J(x_N)}\Bigg|_{J=0}$$

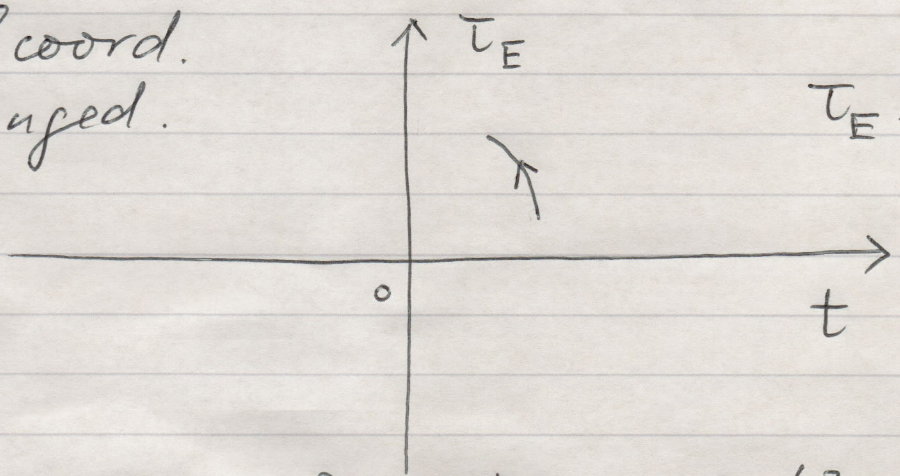
<u>Example</u> <u>Free real scalar field</u>

Euclidean (Wick-rotated) theory

Introduce $\bar{x} \equiv (\bar{x}^0, \bar{\mathbf{x}}) = (i x^0, \bar{\mathbf{x}})$

i.e. $t = -i\tau_E$ (Wick rotation)

Spatial coord. unchanged.



$\tau_E \equiv X^0_E \equiv \bar{X}^0$.

Then: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \mathcal{L}_{int}$

$$\to -\mathcal{L}_E \equiv -\left(\frac{1}{2}\bar{\partial}_\mu\phi\,\bar{\partial}^\mu\phi + \frac{m^2}{2}\phi^2\right) + \mathcal{L}_{int}$$

Here $\bar{\partial}_\mu\phi\,\bar{\partial}^\mu\phi = \left(\frac{\partial\phi}{\partial X^0_E}\right)^2 + \frac{\partial\phi}{\partial X^i_E}\frac{\partial\phi}{\partial X^i_E}$

$$\Rightarrow W_E[J] = N_E\int\mathcal{D}\phi\, e^{-\int d^4\bar{x}\left(\mathcal{L}_E - J(\bar{x})\phi(\bar{x})\right)}$$

$$ds^2_{Mink} = c^2dt^2 - d\vec{x}^2 \to dS^2_E = -\left(dX^{0\,2}_E + d\vec{x}^2_E\right)$$

First, consider $\mathcal{L}_{int} = 0$, $W_E = W^0_E$.

$$W^0_E[J] = N_E\int\mathcal{D}\phi\, e^{-\int d^4x_E\left[\frac{1}{2}\partial^E_\mu\phi\,\partial^E_\mu\phi + \frac{m^2}{2}\phi^2 - J\phi\right]}$$

This is a Gaussian integral.

$$\int dx_1 \cdots dx_n \, e^{-\frac{1}{2}\sum_{ij} x_i K_{ij} x_j + \sum_k J_k x_k} =$$

$$= (\det K)^{-1/2} e^{\frac{1}{2}\sum_{ij} J_i (K^{-1})_{ij} J_j} \qquad (*)$$

Integrating by parts gives

$$\int d^4x_E \, \partial^E_\mu \phi \, \partial^E_\mu \phi = \phi \partial^E_\mu \phi \Big|_{-\infty}^{\infty} - \int \phi \partial^E_\mu \partial^E_\mu \phi \, d^4x_E$$

$$= - \int d^4x_E \, \phi(x_E) \partial_E^2 \phi(x_E),$$

under assumption that ϕ vanishes at ∞.

We can use $(*)$ with $K = -\partial_E^2 + m^2$ as an infinite-dim matrix. Then

$$W_E^0[J] = \underbrace{N_E (\det K)^{-1/2}}_{\text{indep. of } J} e^{\frac{1}{2}\int d^4x_E d^4y_E J(x_E) K^{-1}(x_E - y_E) J(y_E)}$$

Can set $N_E = (\det K)^{1/2}$: $W_E^0[0] = 1$.

Need to find the inverse $K^{-1}(x_E - y_E)$.

In momentum space, K^{-1} is simple because K is simple:

$$K(x_E)\,\phi(x_E) = K(x_E)\int \frac{d^4 p_E}{(2\pi)^4}\, e^{i p_E x_E}\, \phi(p_E) =$$

$$= \int \frac{d^4 p_E}{(2\pi)^4}\,(p_E^2+m^2)\,\phi(p_E)\, e^{i p_E x_E}$$

$$\Rightarrow K(p_E) = p_E^2+m^2 \Rightarrow K^{-1} = \frac{1}{p_E^2+m^2}.$$

Then

$$K^{-1}(x_E-y_E) = \int \frac{d^4 p_E}{(2\pi)^4}\,\frac{e^{i p_E(x_E-y_E)}}{p_E^2+m^2} \quad (**)$$

Note that $K = -\partial_E^2 + m^2$ is translation-inv., i.e. $K(x_E) = K(x_E - y_E)$.

We observe that $K^{-1}(x_E-y_E)$ obeys

$$(-\partial_E^2+m^2)\,K^{-1} = \delta^{(4)}(x_E-y_E)$$

This can be seen from (**) remembering that

$$\delta^{(4)}(x_E - y_E) = \int \frac{d^4 p_E}{(2\pi)^4} e^{i p_E (x_E - y_E)}.$$

Thus, $K^{-1} = \Delta_F^E (x_E - y_E)$: Euclidean Feynman propagator

$$\Rightarrow W_E^0[J] = e^{\frac{1}{2}\int d^4x_E d^4y_E J(x_E)\Delta_F^E(x_E - y_E) J(y_E)}$$

<u>Note</u>: integrating over p_0^E in (**) does not encounter poles (they are now at $p_0^E = \pm i\sqrt{\vec{p}^2 + m^2}$).

Analytic continuation to Mink:

$$\Delta_F^E(i(x^0 - y^0), \vec{x} - \vec{y}) = \left\{ \begin{array}{l} p_E^0 = -i p^0_{MINK} \\ p_E^i = p^i_{MINK} \quad i = 1,2,3 \end{array} \right\}$$

$$= i \int \frac{d^4 p}{(2\pi)^4} e^{+ip(x-y)} \frac{1}{p^2 - m^2} = i \Delta_F(x-y).$$

$$\Rightarrow W_0[J] = e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\,\Delta_F(x-y)\,J(y)}$$

in Mink. space.

<u>Note</u>: Alternative way to compute $W_0[J]$

For $W_E^0[J] = N_E \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int d^4x_E\, \phi[-\partial_E^2 + m^2]\phi + \int d^4x_E\, J\phi}$

make a change of variables: $\phi \to \tilde{\phi} = \phi - \phi_0$, where ϕ_0 satisfies $(-\partial^2 + m^2)\phi_0 = J$.

With $\phi = \phi_0 + \tilde{\phi}$, the expression in the exp. reduces to $-\frac{1}{2}\tilde{\phi}[-\partial_E^2 + m^2]\tilde{\phi} + \frac{1}{2}J\phi_0$.

But $\phi_0 = \int \Delta_F^E(x-y)\,J(y)\,dy$, where

$$(-\partial_E^2 + m^2)\,\Delta_F^E(x_E - y_E) = \delta^{(4)}(x_E - y_E).$$

$$\Rightarrow W_E^0[J] = \underbrace{N_E \int \mathcal{D}\tilde{\phi}\, e^{-\frac{1}{2}\int \tilde{\phi}(-\partial_E^2 + m^2)\tilde{\phi}\, d^4x_E}}_{1} \times e^{\frac{1}{2}\int d^4x_E\, d^4y_E\, J(x_E)\,\Delta_F^E(x_E - y_E)\,J(y_E)},$$

as before

$$G_0^{(N)}(x_1 \cdots x_N) = \frac{1}{i^N} \left. \frac{\delta^N W_0[J]}{\delta J(x_1) \cdots \delta J(x_N)} \right|_{J=0}$$

<u>$N=1$:</u>

$$G_0^{(1)}(x_1) = \frac{1}{i} \left. \frac{\delta W_0[J]}{\delta J(x_1)} \right|_{J=0} =$$

$$= \frac{1}{i}\left(-\frac{i}{2}\right) 2 \int d^4x \, J(x) \Delta_F(x-x_1) e^{-\frac{i}{2} J \Delta J}$$

With $J=0 \Rightarrow G_0^{(1)}(x_1) = 0.$

<u>$N=2$:</u>

$$G_0^{(2)}(x_1, x_2) = \frac{1}{i^2} \left. \frac{\delta^2 W_0[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} =$$

$$= i \Delta_F(x_2 - x_1)$$

More explicitly:

$$\frac{1}{i^2} \frac{\delta^2 W_0[J]}{\delta J(x_1) \delta J(x_2)} = i \Big[\Big(\Delta_F(x_2 - x_1) -$$

$$- i \int d^4x \, J(x) \, \Delta_F(x - x_1) \int d^4x \, J(x) \, \Delta_F(x - x_2) \Big)$$

$$\times e^{-\frac{i}{2} J \Delta J} \Big] .$$

Clearly, $\left. \dfrac{\delta^3 W_0[J]}{\delta J(x_1) \, \delta J(x_2) \, \delta J(x_3)} \right|_{J=0} = 0$

Explicitly:

$$\frac{1}{i^3} \frac{\delta^3 W_0[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3)} = \Big[- i \Delta_F(x_1 - x_3) \int d^4x \, J(x) \, \Delta_F(x - x_3)$$

$$- i \Delta_F(x_1 - x_3) \int d^4x \, J(x) \Delta_F(x - x_2) -$$

$$- i \Delta_F(x_2 - x_3) \int d^4x \, J(x) \, \Delta_F(x - x_1) -$$

$$- \int d^4x \, J(x) \, \Delta_F(x - x_1) \int d^4x \, J(x) \, \Delta_F(x - x_2) \times$$

$$\times \int d^4x \, J(x) \, \Delta_F(x - x_3) \Big] e^{-\frac{i}{2} J \Delta J} .$$

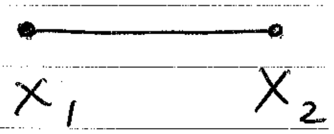
Continuing to $G_0^{(4)}(x_1, x_2, x_3, x_4)$, we obtain (check this!):

$$G_0^{(4)}(x_1 x_2 x_3 x_4) = i^2 \Big[\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4)$$

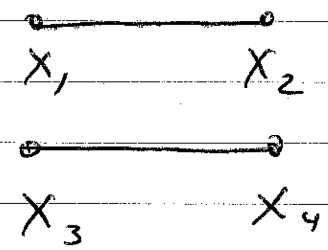
$$+ \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3) \Big]$$

Feynman diagrams:

$$G_0^{(2)}(x_1, x_2) =$$



$$G_0^{(4)}(x_1, x_2, x_3, x_4) =$$



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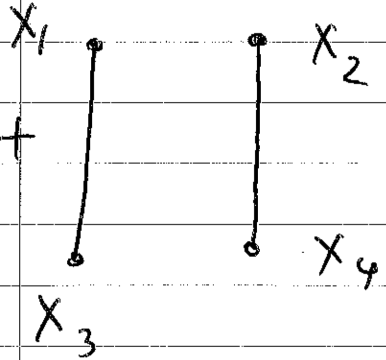

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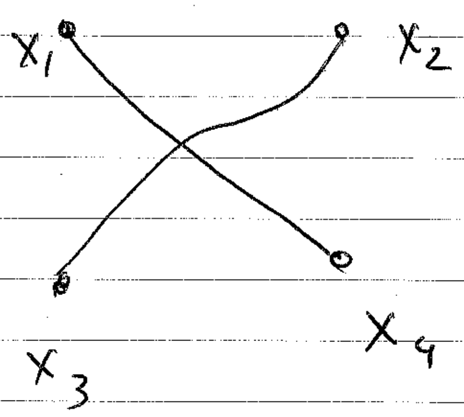


$$\Rightarrow \quad G_0^{(2n+1)}(x_1 \dots x_{2n+1}) = 0$$

$$G_0^{(2n)}(x_1 \cdots x_{2n}) = \sum_{perm} G_0^{(2)}(x_{k_1}, x_{k_2}) \cdots G_0^{(2)}(x_{k_{2n-1}}, x_{k_{2n}})$$

e.g. $G_0^{(6)}$ has 15 terms.

What about $Z[J]$ - ?

$$W[J] = e^{iZ[J]}$$

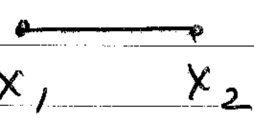
Note: $W[0] = 1 \Rightarrow Z[0] = 0$.

$$G^{(N)}_{\text{connected}} = \frac{1}{i^{N-1}} \left. \frac{\delta^N Z[J]}{\delta J(x_1) \cdots \delta J(x_N)} \right|_{J=0}$$

Since $W_0[J] = e^{-\frac{i}{2}\int d^4x\, d^4y\, J(y) \Delta_F(y-x) J(x)}$

$$Z_0[J] = -\frac{1}{2}\int d^4x\, d^4y\, J(y)\Delta_F(y-x)J(x)$$

$$G^{(2)}_{0\,\text{connected}}(x_1, x_2) = i\,\Delta_F(x_1 - x_2) = G^{(2)}_0(x_1 - x_2)$$

$=$ 

All other $G^{(n)}_{0,\text{connected}}$ vanish since $Z_0[J]$ is only quadratic in J.