<u>Reminder</u>: Green's function of a linear operator $\hat{L}$ obeys

$$\hat{L}_x G(x-y) = \delta(x-y)$$

Then the solution of an inhomogeneous eq

$$\hat{L}\phi = J$$

is written as

$$\phi = \phi_0 + \int G(x-y) J(y)\, dy,$$

where $\hat{L}\phi_0 = 0$.

<u>Note</u>: J may depend on ϕ $\Rightarrow$ then we get an integral eq ϕ (i.e. we convert a non-lin. diff. eq. into an integral eq.). This integral eq. then can be solved perturbatively.

<u>Note</u>: ODE comes with b.c. or init.c. $\Rightarrow$ can have different $G(x-y)$ depending on the choice of b.c. for the same $\hat{L}$.

<u>Optional</u>: you should know Green's

functions of standard $\hat{L}$ (diffusion, Laplace, d'Alembert etc) in $\forall$ dim. (see e.g. Vladimirov, Eq. of math. physics).

Green's function(s) of Klein-Gordon operator

$$\left(\Box_x + m^2 \right) G(x-y) = \delta^{(4)}(x-y)$$

Fourier: $G(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \tilde{G}(p)$

$$\delta^{(4)}(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)}$$

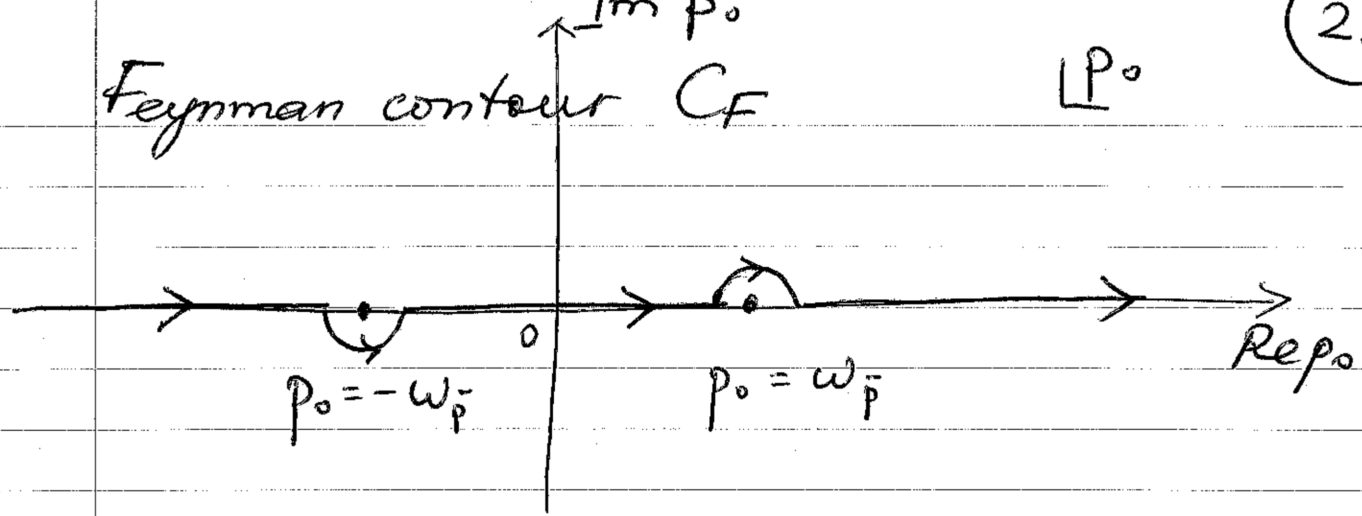
$$\Rightarrow \left(-(p^0)^2 + \vec{p}^{\,2} + m^2 \right) \tilde{G}(p) = 1$$

$$\Rightarrow \tilde{G}(p) = -\frac{1}{p^2 - m^2}$$: Simple poles at $p_0 = \pm\sqrt{\vec{p}^{\,2} + m^2}$

$$G(x-y) = -\int_C \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2}.$$

We get different G by choosing different contours C.

Feynman contour C_F

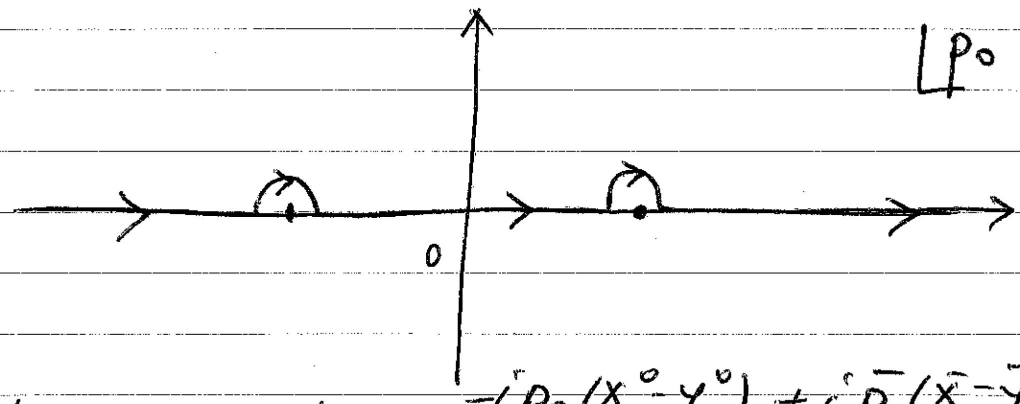


$$G_F(x-y) = -\int_{C_F} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2-m^2} =$$

$$= -\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon} = i\, D_F(x-y)$$

defined earlier.

- Other Green's functions



Looking at $e^{-ip_0(x^0-y^0)+i\bar{p}(\bar{x}-\bar{y})}$:

- $x^0 < y^0$: need to close C in upper half-plane for convergence

$\Rightarrow G(x-y) \sim \theta(x^0-y^0)$

- $x^0 > y^0$: need to close C in lower half-plane

$$G_R(x-y) = -\int_{C_R} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2-m^2} =$$

$$= +2\pi i \left[-\int \frac{d^3p}{(2\pi)^4} \frac{1}{2\omega_p} e^{i\omega_p(x^0-y^0)+i\bar{p}(\bar{x}-\bar{y})} \right.$$

$$\left. + \int \frac{d^3p}{(2\pi)^4} \frac{1}{2\omega_p} e^{-i\omega_p(x^0-y^0)+i\bar{p}(\bar{x}-\bar{y})} \right]$$

$$= i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[e^{ip(x-y)} + e^{-ip(x-y)} \right]$$

$= + i\Delta(x-y)$ for $x^0 > y^0$ and 0 for $x^0 < y^0$.

$$G_R(x-y) = +i\,\theta(x^0-y^0)\,[\phi(x), \phi(y)]$$

Since $G_F(x-y) = i D_F(x-y)$, we can define $G_R = i\Delta_R$, with

$$\Delta_R(x-y) = +\theta(x^0-y^0)\,[\phi(x), \phi(y)]$$