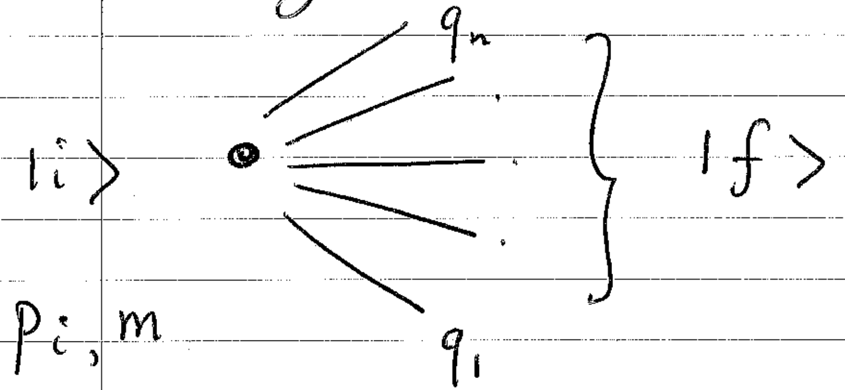
Decay rates



$$p_f = \sum_{j=1}^{n} q_j$$

$$P_n = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle i|i\rangle\langle f|f\rangle}$$

<u>Recall:</u> $\langle i|j\rangle = (2\pi)^3\, 2\omega_{p_i}\, \delta^{(3)}(p_i - p_j)$

$\langle i|i\rangle = \cancel{(2\pi)^3}\, 2\omega_{p_i} V$, since $(2\pi)^3 \delta^{(3)}(0) = V$.

$$\langle f|f\rangle = \prod_{j=1}^{n} 2\omega_{q_j} V$$

$$\langle f|\hat{S} - \mathbb{1}|i\rangle = \langle f|i\hat{T}|i\rangle =$$

$$= i\,(2\pi)^4 \delta^{(4)}(p_f - p_i)\, \mathcal{M}(i \to f)$$

$$P_n = \frac{(2\pi)^4 \delta^{(4)}(p_f - p_i)\, VT\, |\mathcal{M}_f|^2}{2\omega_{p_i} V \prod_{j=1}^{n} 2\omega_{q_j} V}$$

We now sum over final states:

$$\sum_{n_j} \rightarrow \frac{V}{(2\pi)^3}\int d^3 q_j$$

<u>Recall</u>: in finite vol, $q_i = \frac{2\pi}{L} n_i$,

$$\int d^3 q \rightarrow \left(\frac{2\pi}{L}\right)^3 \sum_n$$

This gives the measure:

$$d\Pi_n = (2\pi)^4 \delta^{(4)}(P_f - P_i) \prod_{j=1}^{n} \frac{d^3 q_j}{(2\pi)^3 2\omega_{q_j}}$$

Defining the <u>rate</u> $\Gamma_n = P_n / T$, we get

$$\Gamma_n = \frac{1}{2\omega_{p_i}} \int d\Pi_n \, |\mathcal{M}_{fi}|^2$$

- rate of decay into n-particle state.

Total decay width

$$\Gamma = \frac{1}{2\omega_{p_i}} \sum_n \int d\Pi_n \, |\mathcal{M}_{fi}|^2$$

Half-life $\tau = 1/\Gamma$

If the initial particle is at rest,

$\omega_{p_i} = m$ and

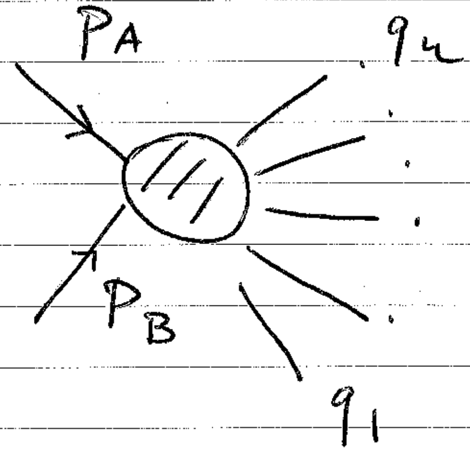
$$\Gamma_{rest} = \frac{1}{2m} \sum_n \int d\Pi_n \, |\mathcal{M}_{fi}|^2$$

$$\Rightarrow \tau = \tau_{rest} \frac{\omega_{p_i}}{m} = \tau_{rest} \frac{\mathcal{E}}{mc^2} = \tau_{rest}\gamma$$

$$\tau = \tau_{rest} / \sqrt{1-\bar{v}^2} > \tau_{rest}.$$

<u>Cross - sections</u>

$|i\rangle = |p_A \, p_B\rangle$



$$|f\rangle = |q_1 \cdots q_n\rangle$$

Trans. prob. per unit time per flux

$$d\sigma = \frac{1}{F} \frac{d\Pi_n}{4\omega_{\bar{A}} \omega_{\bar{B}} V} |\mathcal{M}_{if}|^2$$

(differential cross-section)

Flux $F = \frac{|\bar{v}_{rel}|}{V} = \frac{|\bar{v}_A - \bar{v}_B|}{V} =$

$$= \frac{|\bar{p}_A/\mathcal{E}_A - \bar{p}_B/\mathcal{E}_B|}{V}$$

Here $p_A = (\mathcal{E}_A, \bar{p}_A)$, $p_B = (\mathcal{E}_B, \bar{p}_B)$

$\mathcal{E}_A \equiv \omega_A$, $\mathcal{E}_B \equiv \omega_A$; $\mathcal{E}^2 = \bar{p}^2 + m^2$.

Can show (see p. 36) that

$$F = \frac{\sqrt{(p_A p_B)^2 - m_A^2 m_B^2}}{\mathcal{E}_A \mathcal{E}_B V}$$

$$\Rightarrow d\sigma = \frac{1}{4\sqrt{(p_A p_B)^2 - m_A^2 m_B^2}} |\mathcal{M}_{fi}|^2 d\Pi_n$$

<u>Differential cross-section for 2 → 2 scattering.</u>

In CMF : $\bar{p}_A + \bar{p}_B = 0$

$\Rightarrow \bar{q}_1 + \bar{q}_2 = 0$ in CMF

In detail:

$$\left|\frac{\bar{p}_1}{\varepsilon_1} - \frac{\bar{p}_2}{\varepsilon_2}\right| = \sqrt{\frac{\bar{p}_1^{\,2}}{\varepsilon_1^2} + \frac{\bar{p}_2^{\,2}}{\varepsilon_2^2} - \frac{2\bar{p}_1\bar{p}_2}{\varepsilon_1\varepsilon_2}}$$

$$= \sqrt{\frac{\bar{p}_1^{\,2}\varepsilon_2^2 + \bar{p}_2^{\,2}\varepsilon_1^2 - 2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2}{\varepsilon_1^2\varepsilon_2^2}}.$$

But: $(p_1p_2)^2 - m_1^2m_2^2 = (\varepsilon_1\varepsilon_2 - \bar{p}_1\bar{p}_2)^2 - m_1^2m_2^2$

$$= \varepsilon_1^2\varepsilon_2^2 - 2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2 + \bar{p}_1^{\,2}\bar{p}_2^{\,2} - m_1^2m_2^2 =$$

$$= (\bar{p}_1^{\,2} + m_1^2)(\bar{p}_2^{\,2} + m_2^2) + \bar{p}_1^{\,2}\bar{p}_2^{\,2} - m_1^2m_2^2 -$$

$$- 2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2 =$$

$$= \underbrace{\bar{p}_1^{\,2}\bar{p}_2^{\,2} + m_1^2\bar{p}_2^{\,2}} + \underbrace{\bar{p}_1^{\,2}\bar{p}_2^{\,2} + m_2^2\bar{p}_1^{\,2}} -$$

$$- 2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2 =$$

$$= \varepsilon_1^2\bar{p}_2^{\,2} + \varepsilon_2^2\bar{p}_1^{\,2} - 2\varepsilon_1\varepsilon_2\bar{p}_1\bar{p}_2$$ as above.

$\{ |\bar{p}_A| = |\bar{p}_B| = |\bar{q}_1| = |\bar{q}_2| \underbrace{\text{elastic scattering}} \}$

For simplicity, <u>assume</u> $m_A = m_B = m_1 = m_2 \equiv m$

Since $p_A + p_B = q_1 + q_2$ and $p_A^2 = m^2$

etc $\Rightarrow$ $p_A p_B = q_1 q_2 = \varepsilon_1 \varepsilon_2 - \bar{q}_1 \bar{q}_2$.

Now: $(p_A p_B)^2 - m^4 = (\varepsilon_1 \varepsilon_2 - \bar{q}_1 \bar{q}_2)^2 -$

$- (\varepsilon_1^2 - \bar{q}_1^2)(\varepsilon_2^2 - \bar{q}_2^2) = \left(\text{since } \bar{q}_1 = -\bar{q}_2\right) =$

$= (\varepsilon_1 \varepsilon_2 + \bar{q}_1^2)^2 - (\varepsilon_1^2 - \bar{q}_1^2)(\varepsilon_2^2 - \bar{q}_1^2) =$

$= (\varepsilon_1 + \varepsilon_2)^2 \bar{q}_1^2 = \mathcal{E}_{CMF}^2 \bar{q}_1^2$

$$\mathcal{E}_{CMF} = (\varepsilon_1 + \varepsilon_2)_{CMF}$$

Now consider:

$$\int d\Pi_2 = \int \frac{d^3 q_1}{(2\pi)^3} \frac{1}{2\varepsilon_1} \int \frac{d^3 q_2 \,(2\pi)^4 \delta^{(4)}(p_A + p_B - q_1 - q_2)}{2\varepsilon_2 (2\pi)^3}$$

1) Integrate over $d^3 q_2$ using $\delta^{(3)}(\bar{p}_A + \bar{p}_B - \bar{q}_1 - \bar{q}_2)$

$\Rightarrow$ this just means $\bar{q}_1 = -\bar{q}_2$

$$\int d\Pi_2 = \int \frac{d^3 q_1}{(2\pi)^3} \frac{1}{2\mathcal{E}_1\, 2\mathcal{E}_2} 2\pi\, \delta(\mathcal{E}_{CMF} - \mathcal{E}_1 - \mathcal{E}_2)$$

$$= \int \frac{d\Omega\, d|\bar{q}_1|\, |q_1|^2}{(2\pi)^3 (2\mathcal{E}_1\, 2\mathcal{E}_2)} 2\pi\, \delta(\mathcal{E}_{CMF} - \mathcal{E}_1 - \mathcal{E}_2)$$

Here $\mathcal{E}_1 = \mathcal{E}_2 = \sqrt{\bar{q}_1^{\,2} + m^2}$.

<u>Recall</u>: $\delta(F(x)) = \sum_i \frac{\delta(x - x_i^0)}{|F'(x_i^0)|}$,

where x_i^0 are zeros of F: $F(x_i^0) = 0$.

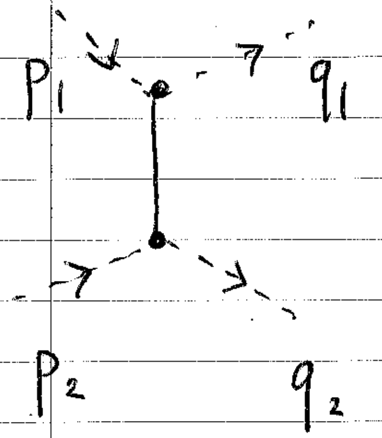
$$\Rightarrow \int d\Pi_2 = \frac{1}{16\pi^2} \int d\Omega\, \frac{\bar{q}_1^{\,2}\, d|q_1|}{\mathcal{E}_1^2}\, \delta\left(\mathcal{E}_{CMF} - 2\sqrt{\bar{q}_1^{\,2} + m^2}\right)$$

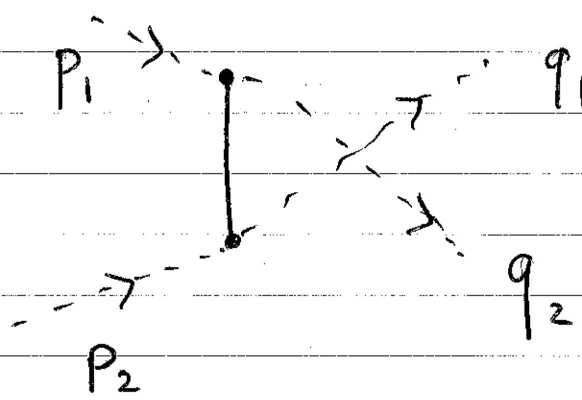
$$= \frac{1}{16\pi^2} \int d\Omega\, \frac{|q_1|}{2\mathcal{E}_1} = \frac{1}{16\pi^2} \int d\Omega\, \frac{|q_1|}{\mathcal{E}_{CMF}} .$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|\mathcal{M}_{fi}|^2}{\mathcal{E}_{CMF}^2} = \frac{1}{64\pi^2} \frac{|\mathcal{M}_{fi}(s,t,u)|^2}{s}}$$

Examples:

- scalar Yukawa theory at $O(g^2)$:





$$\mathcal{M}_{fi} = (-ig)^2 \left[\frac{1}{t - m^2 + i\varepsilon} + \frac{1}{u - m^2 + i\varepsilon} \right]$$

$$t = (p_1 - q_1)^2 \qquad u = (p_1 - q_2)^2$$

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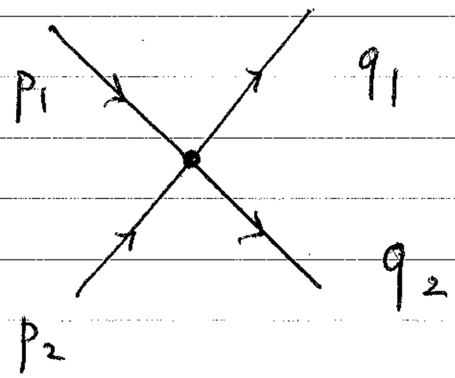
- $\lambda\phi^4$ theory $\left(\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4 \right)$.

At $O(\lambda)$:



$$\mathcal{M}_{fi} = -i\lambda + O(\lambda^2)$$

$$d\sigma = \frac{\lambda^2}{64\pi^2 E_{CMF}^2} d\Omega \qquad d\Omega = \sin\theta \, d\theta \, d\varphi .$$

Scalar Yukawa theory (continued):

$$t = (p_1 - q_1)^2 \qquad \underline{CMF}: \ \bar{p}_1 + \bar{p}_2 = 0$$

$$\bar{q}_1 + \bar{q}_2 = 0$$

$$u = (p_1 - q_2)^2 \qquad |\bar{p}_1| = |\bar{q}_1|$$

$$p_1 = \left(\sqrt{\bar{p}_1^{\,2} + M^2}, \ \bar{p}_1 \right)$$

$$q_1 = \left(\sqrt{\bar{q}_1^{\,2} + M^2}, \ \bar{q}_1 \right)$$

If $\bar{q} \equiv \bar{p}_1 - \bar{q}_1$, $\bar{q}_* = \bar{p}_1 - \bar{q}_2$,

$$t = -(\bar{p}_1 - \bar{q}_1)^2 \equiv -\bar{q}^{\,2}$$

$$u = -\bar{q}_*^{\,2} \quad \Rightarrow$$

$$\mathcal{M}_{fi} = g^2 \left[\frac{1}{\bar{q}^{\,2} + m^2} + \frac{1}{\bar{q}_*^{\,2} + m^2} \right].$$

In QM, with $H = \dfrac{\bar{p}_1^{\,2}}{2M} + \dfrac{\bar{p}_2^{\,2}}{2M} + V(|\bar{r}_2 - \bar{r}_1|)$,

$$V(r) = g^2 \frac{e^{-mr}}{r} \Rightarrow \mathcal{M}_{Born} \sim \langle f | V | i \rangle$$

$$\mathcal{M}_{Born} \sim \int d^3x\, e^{-i\bar{q}\bar{r}} g^2 \frac{e^{-mr}}{r} \sim$$

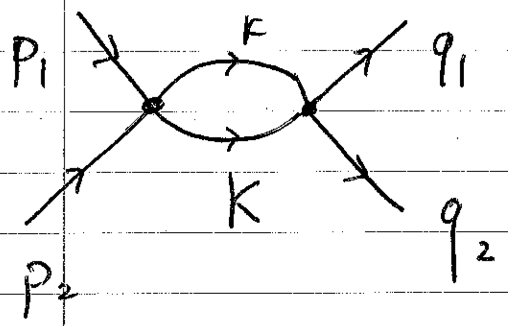
$$\sim g^2 \frac{1}{\bar{q}^2 + m^2}$$ (Do the explicit computation !)

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - M^2 \varphi^* \varphi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - g \varphi^* \varphi \phi$$

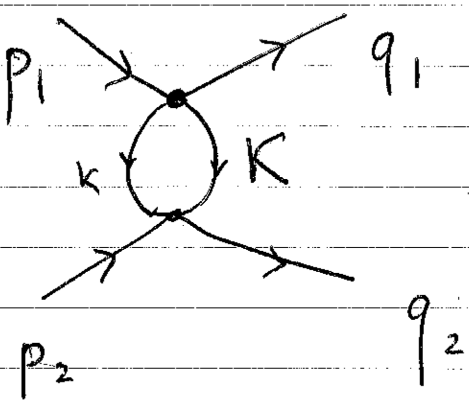
We have interaction via exchange by scalar ϕ of mass m.

Note: the term $\frac{1}{q_*^2 + m^2}$ from the u-channel arises from the exchange interaction in QM (final states are indistinguishable, must use symmetrized $\psi = \frac{1}{\sqrt{2}} \left(\varphi_1(r_1) \varphi_2(r_2) + \varphi_1(r_2) \varphi_1(r_1) \right)$).

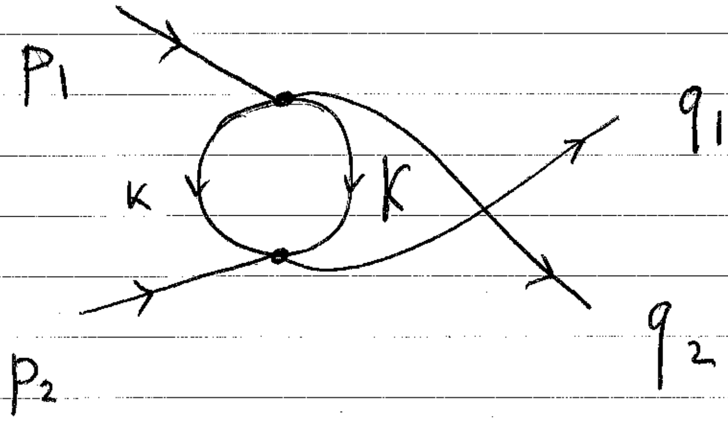
At $O(\lambda^2)$:



$p_1 + p_2 = k + K$



$p_1 - q_1 = k + K$



$p_1 - q_2 = k + K$

Each diagram contributes to $\mathcal{M}_{fi}$:

$$\frac{(-i\lambda)^2}{2!}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{K^2-m^2+i\varepsilon}$$

At $|k| \to \infty$ limit: $\int \frac{d^4k}{k^4} \sim \int \frac{dk}{k} \sim$

$\sim \ln|K| \to \infty$ Disaster.

Introduce cutoff at $|k| = \Lambda$.

Then:

$$\mathcal{M}_{fi} = -i\lambda + iC\lambda^2 \left[\ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{t} + \ln \frac{\Lambda^2}{u} \right] + O(\lambda^3)$$

(For details of evaluating integrals see A. Zee, QFT in Nutshell, III. 2)

Here C is a constant and we put $m = 0$ for simplicity. Now let:

$$-i\lambda_{phys_0} \equiv -i\lambda + iC\lambda^2 L_0 + \dots \qquad (*)$$

$$L_0 \equiv \ln \frac{\Lambda^2}{s_0} + \ln \frac{\Lambda^2}{t_0} + \ln \frac{\Lambda^2}{u_0}.$$

$$-i\lambda_{phys_1} = -i\lambda + iC\lambda^2 L_1 + \dots \qquad (**)$$

$$L_1 \equiv \ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{t} + \ln \frac{\Lambda^2}{u}$$

Solve $(*)$ for λ perturbatively.

$$\lambda = a_1 \lambda_{p_0} + a_2 \lambda_{p_0}^2 + \dots$$

$$-i\lambda_{p_0} = -i a_1 \lambda_{p_0} - i a_2 \lambda_{p_0}^2 + \dots +$$

$$+ i C \lambda_{p_0}^2 a_1^2 L_0 + O(\lambda_{p_0}^3)$$

$$\Rightarrow a_1 = 1 \qquad a_2 = C L_0$$

$$\Rightarrow \lambda = \lambda_{p_0} + C L_0 \lambda_{p_0}^2 + \dots$$

Now subst. into $\lambda_{p_1} = \dots$

$$\lambda_{p_1} = \lambda_{p_0} + C L_0 \lambda_{p_0}^2 - C L_1 \lambda_{p_0}^2 + \dots$$

$$\Rightarrow \boxed{\lambda_{p_1} = \lambda_{p_0} + \lambda_{p_0}^2 C \left[\ln \frac{s_1}{s_0} + \ln \frac{t_1}{t_0} + \ln \frac{u_1}{u_0} \right] + O(\lambda_{p_0}^3)}$$

- <u>Remark</u>: λ_{phys} (or $\mathcal{M}_{fi}^{phys}$) should be independent of Λ $\Rightarrow$

$$\frac{d\,\lambda_{phys}}{d\ln\Lambda} = 0 \qquad \left(d\ln\Lambda = \frac{d\Lambda}{\Lambda} \right)$$

$$\frac{d}{d\ln\Lambda}\left(\mathcal{M}_{fi}\right) = \frac{d}{d\ln\Lambda}\left[-i\lambda + ic\lambda^2\left(3\ln\Lambda^2 - \ln s t u\right)\right] + O(\lambda^3)$$

$$-i\frac{d\lambda}{d\ln\Lambda} + ic\,2\lambda\frac{d\lambda}{d\ln\Lambda}\left(\cdots\right) + ic\lambda^2\cdot 6 = 0$$

$$\sim O(\lambda^3)$$

$$\Rightarrow \boxed{\Lambda\frac{d\lambda}{d\Lambda} = 6c\lambda^2 + O(\lambda^3)}$$

$$\Rightarrow \lambda = \lambda(\Lambda).$$