Pauli-Jordan function

(13)

Microcausality

$$\Delta(x-y) = [\phi(x), \phi(y)] \quad \text{(Pauli-Jordan function)}$$

$$\Delta(x-y) = \left[\int d^3\tilde{p} \left(a_p e^{-ipx} + a_p^+ e^{ipx} \right), \int d^3\tilde{q} \left(a_q e^{-iqy} + a_q^+ e^{iqy} \right) \right] =$$

$$= \int d^3\tilde{p}\, d^3\tilde{q} \left([a_p a_q^+] e^{-ipx+iqy} + [a_p^+ a_q] \times e^{ipx-iqy} \right)$$

$$= \int d^3\tilde{p}\, d^3\tilde{q} \left(e^{-ipx+iqy} - e^{ipx-iqy} \right) (2\pi)^3 2\omega_p \, \delta(\bar{p}-\bar{q}) =$$

$$= \int d^3\tilde{p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right)$$

Here $\int d^3\tilde{p} \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}$.

Now recall that

$$\delta(x^2-a^2) = \frac{1}{2|a|} \left(\delta(x-a) + \delta(x+a) \right).$$

$$\delta(p^2-m^2)=\frac{1}{2\omega_p}\left[\delta(p_0+\omega_p)+\delta(p_0-\omega_p)\right]$$

Then

$$\Delta(x-y)=\int dp_0 \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[\delta(p_0+\omega_p)+\right.$$

$$\left.+\,\delta(p_0-\omega_p)\right]\varepsilon(p_0)\,e^{-ip(x-y)}$$

Here $\varepsilon(p_0)=\frac{p_0}{|p_0|}=\operatorname{sgn}p_0$.

$$\Delta(x-y)=\int\frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\,\varepsilon(p_0)\,e^{-ip(x-y)}$$

<u>This expression is Lorentz-invariant.</u>

- The sign of A^0 component of a time-like 4-vector A^μ is invar. under $\Lambda\in L^\uparrow$ (<u>orthochronous comp.</u> of Lor. group).

$\Lambda^0{}_0\geq 1$

Indeed, $A'^\mu=\Lambda^\mu{}_\nu A^\nu$

$A'^0=\Lambda^0{}_0A^0+\Lambda^0{}_iA^i$

To show that the sign of A^0 is preserved, note that $\Lambda^0{}_0 > 0$ for $\Lambda \in L^\uparrow$,

and estimate $\Lambda^0{}_i A^i$:

We have (Cauchy - Bunyakovsky):

$$(\Lambda^0{}_i A^i)^2 \leq (\Lambda^0{}_i \Lambda^0{}_i)(A_i A_i)$$

Also: $(A^0)^2 - \bar{A}^2 > 0$ (A^μ is time-like)

And: $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$

$$\Rightarrow (\Lambda^0{}_0)^2 - (\Lambda^0{}_i)^2 = 1$$

So $(\Lambda^0{}_i A^i)^2 \leq \left[(\Lambda^0{}_0)^2 - 1\right] \bar{A}^2 <$

$$< \left[(\Lambda^0{}_0)^2 - 1\right](A^0)^2 < (\Lambda^0{}_0)^2 (A^0)^2$$

$$\Rightarrow |\Lambda^0{}_i A^i| < |\Lambda^0{}_0| |A^0|,$$

i.e. the term $\Lambda^0{}_i A^i$ will not

affect the sign of $\Lambda^0{}_0 A^0$.

We note that $\Delta(x-y) = 0$ at equal

times for all $(x-y)^2 = -(\bar{x}-\bar{y})^2 < 0$

$$[\phi(t,\bar{x}), \phi(t,\bar{y})] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[e^{i\bar{p}(\bar{x}-\bar{y})} - e^{-i\bar{p}(\bar{x}-\bar{y})} \right]$$

$= 0$ (changing var. $\bar{p} \to -\bar{p}$ in one of the parts).

(This must be so for equal-time comm. rel.)

Now, $\Delta(x-y)$ is Lor.-inv. and thus is a function of $(x-y)^2$ and $\varepsilon(x^0)$. For space-like separated x, y, $(x-y)^2 < 0$, one can always find $\Lambda \subset L^\uparrow$ to have $(\Lambda^\mu{}_\nu (x-y)^\nu)_0 = 0$, and thus

$$\Delta(x-y) = 0 \quad \forall \; (x-y)^2 < 0.$$

This is an example of microcausality

$$[\hat{O}(x)\hat{O}(y)] = 0 \quad \forall \; (x-y)^2 < 0$$

shown here for a <u>free</u> theory.

In general, one of Wightman axioms of QFT.

- Note: for free fields, $\Delta(x-y)$ is a c-number (complex number, not an operator), so

$$\Delta(x-y) = [\phi(x)\,\phi(y)] = \langle 0|[\phi(x),\phi(y)]|0\rangle$$

- One can compute the momentum integrals and show that

$$i\Delta(x) = \frac{1}{2\pi}\,\varepsilon(x^0)\,\delta(x^2) - \frac{m}{4\pi\sqrt{x^2}}\,\varepsilon(x^0)\,\theta(x^2) \times J_1(m\sqrt{x^2})$$

Here $x^2 = (x^0)^2 - \bar{x}^2$

$$\theta(\alpha) = \begin{cases} 1, & \alpha > 0 \\ 0, & \alpha < 0 \end{cases}$$

$$\varepsilon(\alpha) = \mathrm{sgn}(\alpha) = \begin{cases} 1, & \alpha > 0 \\ -1, & \alpha < 0 \end{cases}$$

Feynman propagator

$$D_F(x-y) = \langle 0 | T\,\phi(x)\phi(y) | 0 \rangle =$$

$$= \theta(x^0 - y^0)\langle 0 | \phi(x)\phi(y) | 0 \rangle +$$

$$+ \theta(y^0 - x^0)\langle 0 | \phi(y)\phi(x) | 0 \rangle$$

$\theta(x)$: Heaviside function

$$T\,\phi(x_1)\cdots\phi(x_n) = \phi(t_{i_n}\, x_{i_n})\cdots\phi(t_{i_1}\, x_{i_1})$$

$$\text{for } t_{i_1} \leq \cdots t_{i_n}$$

(time-ordering)

- Consider $x^0 > y^0$ first:

$$D_F(x-y) = \langle 0 | \phi(x)\phi(y) | 0 \rangle =$$

$$= \langle 0 | \int \tilde{d^3p}\, \tilde{d^3q} \left(a_p e^{-ipx} + a_p^+ e^{ipx}\right)\left(a_q e^{-iqy} + \right.$$

$$\left. + a_q^+ e^{iqy}\right) | 0 \rangle =$$

$$= \langle 0 | \int \tilde{d^3p}\, \tilde{d^3q} \left[a_p\, a_q^+\right] e^{-ipx + iqy} | 0 \rangle =$$

$$= \langle 0 | \tilde{d^3p}\, e^{-ip(x-y)} | 0 \rangle \equiv D(x-y)$$

- Now consider $x^0 < y^0$

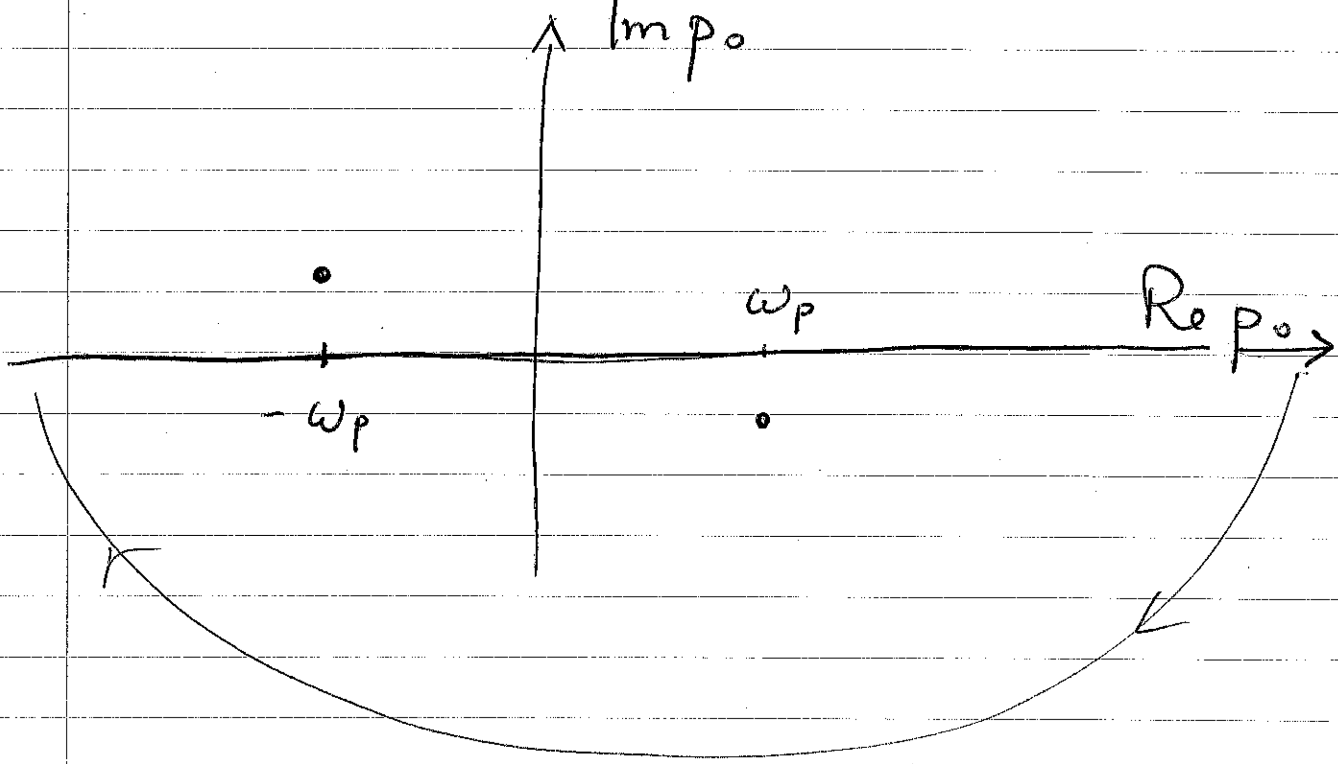
$\Rightarrow D_F(x-y) = D(y-x)$

- $$D_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \Big[\theta(x^0-y^0) e^{-i\omega_p (x^0-y^0)}$$

$$+ \theta(y^0-x^0) e^{i\omega_p (x^0-y^0)} \Big] e^{i\vec{p}(\vec{x}-\vec{y})} =$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p_0-\omega_p+i\varepsilon)(p_0+\omega_p-i\varepsilon)} e^{-ip(x-y)} =$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip(x-y)}$$



$x^0 > y^0$: $e^{-ip_0(x^0-y^0)}$ $\Rightarrow$

if $\mathrm{Im}\, p_0 < 0$, the integrand $\to 0$ on the arc with $|p_0| \to \infty$ $\Rightarrow$ the original integral is equiv. to the one along the closed contour shown in Figure and can be evaluated via Cauchy theorem, picking a pole at $p_0 = \omega_p - i\varepsilon$.

$x^0 < y^0$: now close the contour in the upper half-plane, to have $\mathrm{Im}\, p_0 > 0$.

• Note: $(\Box + m^2) D_F(x-y) =$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} (-p^2 + m^2) e^{-ip(x-y)} =$$

$$= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} = -i\delta^{(4)}(x-y)$$

$\Rightarrow$ $D_F(x-y)$ is a Green's function of the Klein-Gordon eq.