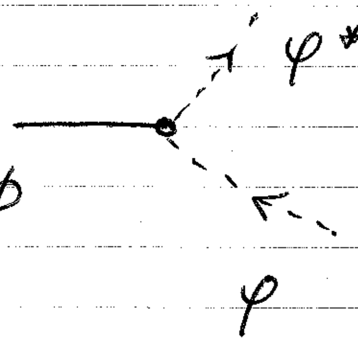
Example: scalar Yukawa theory

$$\mathcal{H}_I(x) = g\,\varphi^*(x)\,\varphi(x)\,\phi(x)$$



Scattering $2 \to 2$: $\quad NN \to NN$

$$|i\rangle = a_+^\dagger(p_1)\, a_+^\dagger(p_2)\, |0\rangle = |p_1, p_2\rangle$$

$$|f\rangle = a_+^\dagger(q_1)\, a_+^\dagger(q_2)\, |0\rangle = |q_1, q_2\rangle$$

Note: can also consider $\bar{N}N \to \bar{N}N$ or $\bar{N}\bar{N} \to \bar{N}\bar{N}$ (charge is conserved)

Amplitudes with $NN \to \bar{N}\bar{N}$ etc do not appear.

$O(g)$ contribution vanishes:

$$\langle 0 |\, T\left(a_{q}\, a_{q_2}\, \varphi^*\varphi\,\phi\, a_{p_1}^\dagger a_{p_2}^\dagger \right) | 0 \rangle$$

unpaired $\Rightarrow 0$.

Convenient to consider $S = 1 + iT$

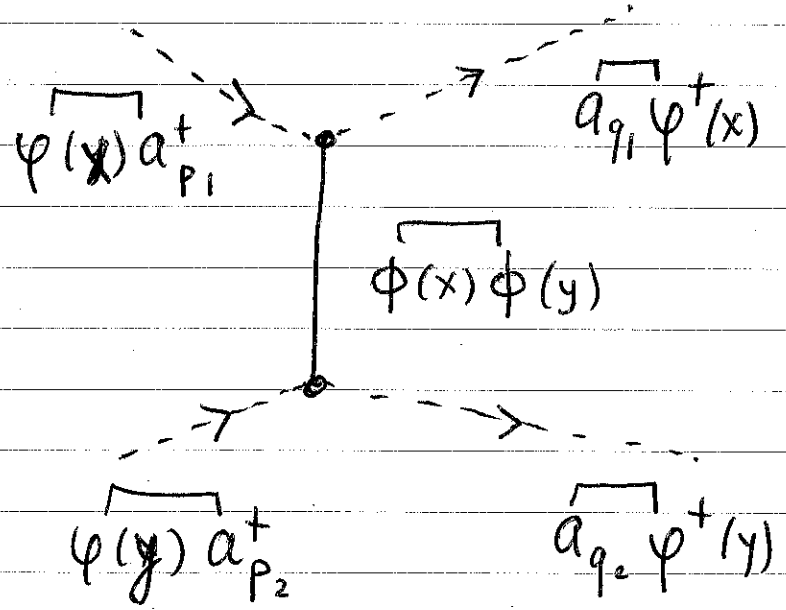
$O(g^2)$ contribution to iT :

$$T_2 = \frac{(-ig)^2}{2}\int d^4x\, d^4y\; T\left(\varphi^\dagger(x)\varphi(x)\phi(x)\,\varphi^\dagger(y)\varphi(y)\phi(y)\right)$$

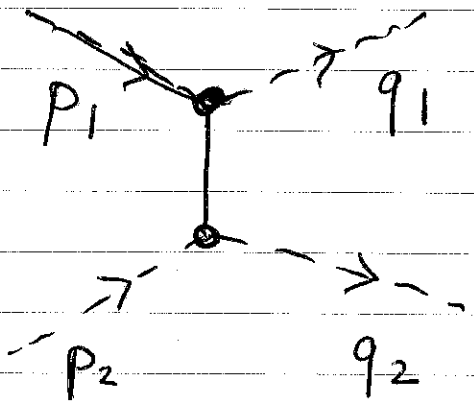
We have:

$$\langle 0|\, T\, a_{q_1} a_{q_2}\, \varphi^\dagger(x)\varphi(x)\phi(x)\,\varphi^\dagger(y)\,\varphi(y)\phi(y)\, a^\dagger_{p_2} a^\dagger_{p_1} |0\rangle$$

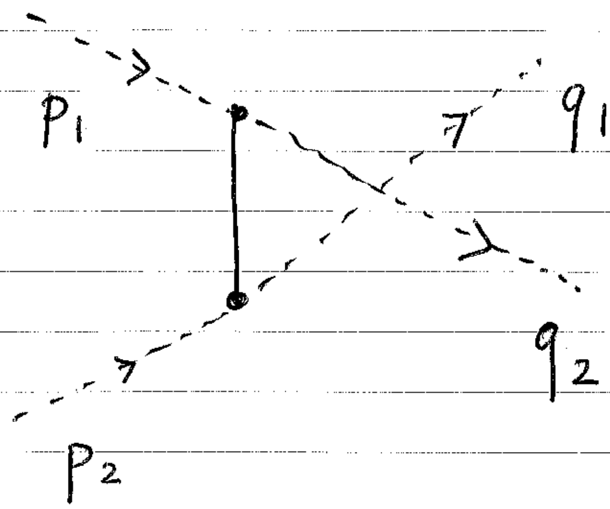
__Diagram:__



__Simpler notation:__



Another diagram:



Algebraically:

$$D_F(x-y) :\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y):$$

$$\langle q_1 q_2 | :\varphi^+(x)\varphi(x)\varphi^+(y)\varphi(y): | p_1 p_2 \rangle$$

$$= \langle q_1 q_2 | \varphi^+(x)\varphi^+(y) | 0 \rangle \langle 0 | \varphi(x)\varphi(y) | p_1 p_2 \rangle$$

$$= e^{i[(q_1-p_1)x+(q_2-p_2)y]} +$$

$$+ e^{i[(q_2-p_1)x+(q_1-p_2)y]} +$$

$$+ e^{i[(q_1-p_1)y+(q_2-p_2)x]} +$$

$$+ e^{i[(q_2-p_1)y+(q_1-p_2)x]}$$

Indeed, recall that

$$\langle 0|\varphi(x)|p\rangle =$$

$$= \langle 0|\int d^3\tilde{q}\left(a_+(q)e^{-iqx} + a_-^+(q)e^{iqx}\right)a_+^+(p)|0\rangle$$

$$= \langle 0|\int d^3\tilde{q}\,[a_+(q)\,a_+^+(p)]\,|0\rangle e^{-iqx} =$$

$$= \langle 0|\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}(2\pi)^3 2\omega_p\,\delta^{(3)}(\bar{p}-\bar{q})\,e^{-iqx}|0\rangle$$

$$= e^{-ipx}.$$

$$T_2 = \frac{(-ig)^2}{2}\int d^4x\,d^4y\int\frac{d^4k}{(2\pi)^4}\frac{i\,e^{ik(x-y)}}{k^2-m^2+i\varepsilon}\times$$

$$\times\left(e^{i[(q_1-p_1)x+(q_2-p_2)y]} + e^{i[(q_2-p_1)x+(q_1-p_2)y]}\right.$$

$$\left. + \{x\leftrightarrow y\}\right)$$

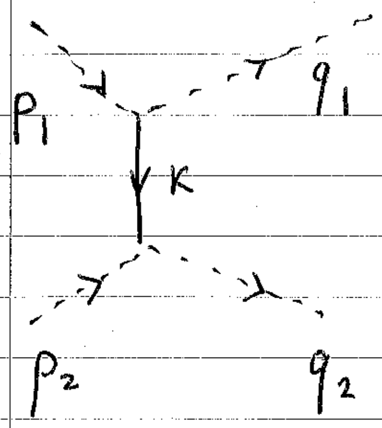
$$T_2 = (-ig)^2 (2\pi)^4 \int d^4k \; \frac{i}{k^2 - m^2 + i\varepsilon} \times$$

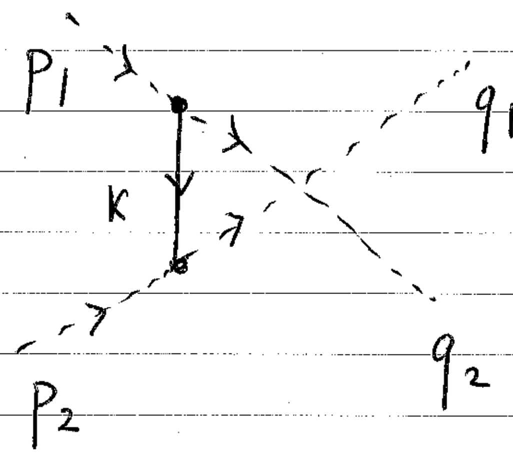
$$\times \Big[\delta^{(4)}(q_1 - p_1 + k)\, \delta^{(4)}(q_2 - p_2 - k) +$$

$$+ \delta^{(4)}(q_2 - p_1 + k)\, \delta^{(4)}(q_1 - p_2 - k) \Big] =$$

$$= i(-ig)^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \times$$

$$\times \left[\frac{1}{(p_1 - q_1)^2 - m^2 + i\varepsilon} + \frac{1}{(p_1 - q_2)^2 - m^2 + i\varepsilon} \right].$$





$p_1 = k + q_1$ $\qquad$ $p_1 = k + q_2$

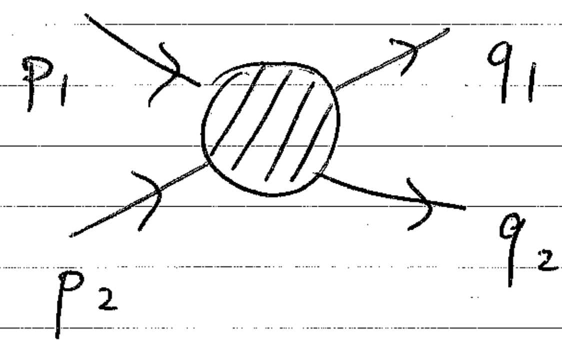
$p_2 + k = q_2$ $\qquad$ $p_2 + k = q_1$

$$S = \mathbb{1} + iT$$

$$\langle f | S - \mathbb{1} | i \rangle = i \langle f | T | i \rangle \equiv$$

$$\equiv i (2\pi)^4 \delta(p_f - p_i) \mathcal{M}(i \to f)$$

Mandelstam variables:



$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2$$

$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2$$

$$u = (p_1 - q_2)^2 = (p_2 - q_1)^2$$

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

Our example involved t and u channels.

s, t, u can be expressed via E_{CMF} and scattering angles => phys. data.

General scheme:

$$\langle p_1 \cdots p_n | \hat{S} | k_1 \cdots k_m \rangle \; - ?$$

With $\hat{S} = \mathbb{1} + i\hat{T}$,

$$\langle p_1 \cdots p_n | i\hat{T} | k_1 \cdots k_m \rangle \; - ?$$

with no initial and final momenta coinciding (no particle is a spectator).

LSZ (Lehmann - Symanzik - Zimmermann) reduction formula

$$\langle p_1 \cdots p_n | i\hat{T} | k_1 \cdots k_m \rangle =$$

$$= \left(i Z^{-1/2}\right)^{n+m} \int d^4x_1 \cdots d^4x_m \int d^4y_1 \cdots d^4y_n \times$$

$$\times e^{ip_1 y_1 + \cdots ip_n y_n - ik_1 x_1 - \cdots - ik_m x_m}$$

$$\times \left(\Box_{x_1} + m^2\right) \cdots \left(\Box_{y_n} + m^2\right) \langle 0 | T\left(\phi(x_1) \cdots \phi(y_n)\right) | 0 \rangle$$

Here: $\phi(x) \to Z^{1/2} \phi_{in}(x) \quad t \to -\infty$

$\phi(x) \to Z^{1/2} \phi_{out}(x) \quad t \to +\infty$

Z is known as wave-function renormalization.

(See e.g. Maggiore „A modern Intro to QFT", § 5.2.)

$\Rightarrow$ Attention shifts to N-point correlation functions

$${}_H\langle 0 | T\, \phi_H(x_1) \cdots \phi_H(x_N) \,|\, 0 \rangle_H \equiv G^{(N)}(x_1 \cdots x_N)$$

(time-ordered products of <u>interacting fields</u>)

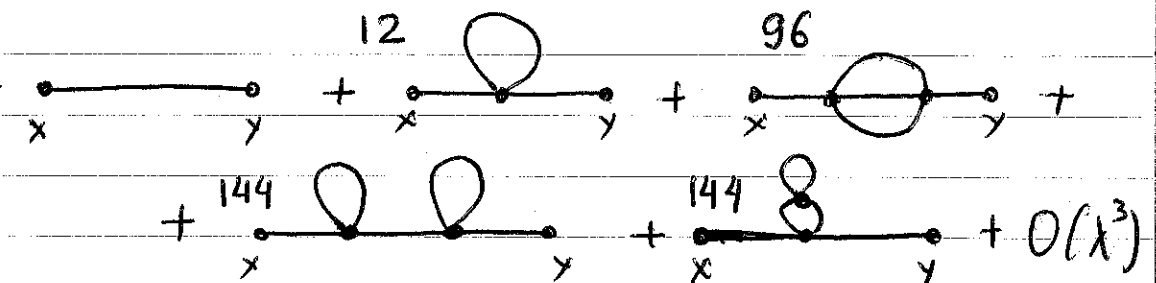
$${}_H\langle 0 | T \phi_H(x_1) \cdots \phi_H(x_N) | 0 \rangle_H =$$

$$= \frac{\langle 0 | T \phi_{ip}(x_1) \cdots \phi_{ip}(x_N)\, e^{-i\int d^4x\, \mathcal{H}^{ip}_{int}} | 0 \rangle}{\langle 0 | T e^{-i\int d^4x\, \mathcal{H}^{ip}_{int}} | 0 \rangle}$$

Gell-Mann and Low, 1951

E.g. in $\lambda\phi^4$ theory:

$G^{(2)}(x,y) =$



How to compute

$$(\Box_x + m^2)\langle 0|T\phi(x)\phi(y)|0\rangle \; ?$$

$$\Box_x = \partial_\mu \partial^\mu = \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{\partial x_i^2}$$

Recall that: $T\phi(x)\phi(y) = \phi(x)\phi(y)\,\Theta(x^0-y^0)$

$+\,\phi(y)\phi(x)\,\Theta(y^0-x^0)$

Also, recall that $\Theta'(x) = \delta(x)$; $\delta(-x) = \delta(x)$

$$\partial_\mu\big(T\phi(x)\phi(y)\big) = T\big(\partial_\mu\phi(x)\phi(y)\big) + [\phi(x)\phi(y)] \times \delta(x^0-y^0)$$

equal-time correlator of fields vanishes

$$\partial^\mu\partial_\mu\big(T\phi(x)\phi(y)\big) = \partial^\mu\{\partial_\mu\phi(x)\phi(y)\,\Theta(x^0-y^0) +$$

$$+\,\phi(y)\partial_\mu\phi(x)\,\Theta(y^0-x^0)\} =$$

$$= T[\partial_\mu\partial^\mu\phi(x)\phi(y)] + \underbrace{[\partial_0\phi(x)\phi(y)]}_{-i\delta^{(3)}(\bar{x}-\bar{y})}\delta(x^0-y^0)$$

$$\Rightarrow \Box_x\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\,\Box_x\phi(x)\phi(y)|0\rangle +$$

$$+\,(-i\delta^{(4)}(x-y)).$$