Then $\mathcal{E}_0 = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2}$

The integral is UV-divergent.
Introduce cutoff at energy scale E_*.
Then $\mathcal{E}_0 \sim E_*^4$ (note that $\mathcal{E}_0$ is energy <u>density</u>, i.e. $\mathcal{E}_0 \sim E/L^3 \sim M^4$)

Vac. energy couples to gravity:

$$\mathcal{L}_{grav} = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \, (R - 2\Lambda)$$

cosmol. const

C.c. term $\sim \frac{1}{8\pi G} \int d^4x \, \Lambda$

Note: $[\Lambda] = 1/L^2 \quad [G] = L^2$

$\Rightarrow \left[\frac{\Lambda}{8\pi G}\right] = M^4$: energy density of vac.

e.g. $\int \sqrt{-g} \, V_{min}(\phi) \, d^4x \sim \int d^4x \, \mathcal{E}_0^4$

$\Rightarrow \mathcal{E}_0 = \left(\Lambda / 8\pi G\right)^{1/4} \sim 10^{-3} \, eV$

with $\Lambda \simeq 4.33 \cdot 10^{-66} \, eV^2$ (PDG)

$E_* \sim 10^{-3}\,eV$ (but QFT works fine at $E \sim 10\,TeV$) ?

On the other hand, imposing cutoff at $E_* \sim 10\,TeV$ or $E_P \sim 10^{19}\,GeV$ overpredicts Λ by orders of magnitude.

Solution for now: ignore gravity, measure the difference

$$:H: \; = H - E_0 = H - \langle 0|H|0\rangle$$

One can view this as the ordering problem: terms $p \cdot q$ and $q p$ are classically equiv. but not in QM.

E.g. $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$ and

$\hat{H} = \frac{1}{2}(\omega\hat{q} - i\hat{p})(\omega\hat{q} + i\hat{p})$ are equiv. classically but in QM give

$$H = \omega(a^\dagger a + 1/2) \quad \text{and}$$

$$H = \omega a^\dagger a, \quad \text{resp.},$$

with the usual def. of a, $a^\dagger$.

Normal ordering:

for $\phi \sim \hat{a} e^{-i\omega t} + \hat{a}^{+} e^{i\omega t}$

$$\sim \phi_+ + \phi_-$$

and $\chi \sim \chi_+ + \chi_-$ define

$$:\phi\chi: = \phi_-\chi_- + \phi_-\chi_+ + \chi_-\phi_+ + \phi_+\chi_+$$

so a^+ always stand to the left of a.

Then $:H: = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \omega_{\vec{p}}\, a^+_{\vec{p}} a_{\vec{p}}$

<u>Note</u>: the <u>difference</u> in vac. energy can be measured, e.g. in

- Casimir effect (see e.g. Itzykson-Zuber, Sect. 3.2.4)
- Spont. symm. breaking

- Single-particle states

$$|p\rangle = a^+_{\bar{p}}|0\rangle$$

We have

$$\langle p|q\rangle = \langle 0|a_{\bar{p}}a^+_{\bar{q}}|0\rangle =$$

$$= \langle 0|\underbrace{(a_{\bar{p}}a^+_{\bar{q}} - a^+_{\bar{q}}}_{[a_{\bar{p}}a^+_{\bar{q}}]}a_{\bar{p}})|0\rangle = (2\pi)^3 2\omega_{\bar{p}}\delta^{(3)}(\bar{p}-\bar{q})$$

$$a_{\bar{p}}|0\rangle = 0$$

Note: $[H, a_{\bar{p}}] = -\omega_{\bar{p}}\, a_{\bar{p}}$

$$[H, a^+_{\bar{p}}] = \omega_{\bar{p}}\, a^+_{\bar{p}}$$

Here we use notations $:H: \equiv H$.

Then $\quad H|p\rangle = \omega_{\bar{p}}|p\rangle$

This quantum state corresp. to one real scalar with $E_p = \sqrt{\bar{p}^2 + m^2} = \omega_{\bar{p}}$.

- Recall that via Noether th. we can construct conserved charge $\bar{P}$ (momentum) for our $\mathcal{L}$.

<u>Exercise:</u> show that

$$\bar{P} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\bar{p}}} \bar{p}\, a^+_{\bar{p}} a_{\bar{p}}$$

is the operator of momentum with

$$\bar{P}\,|p\rangle = \bar{p}\,|p\rangle .$$

- <u>Multi-particle states</u>

$$|p_1, \dots p_n\rangle = a^+_{p_1} \dots a^+_{p_n} |0\rangle$$

– a state with n particles with momenta $p_1 \dots p_n$.

Note that $|p_1, p_2\rangle = a^+_{p_1} a^+_{p_2} |0\rangle =$
$= a^+_{p_2} a^+_{p_1} |0\rangle = |p_2, p_1\rangle$

i.e. <u>the state</u> is <u>symmetric</u> under exchange of particles (particles are bosons) – this follows from

$$[a^+_p\, a^+_q] = 0$$ (specific type of commut. rel.)

Hilbert space: $|0\rangle$, $a^+_{\bar{p}}|0\rangle$, $a^+_{\bar{p}} a^+_{\bar{q}}|0\rangle$ ...

i.e. $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ Fock space

One can introduce the number operator

$$N = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} a^+_{\bar{p}} a_{\bar{p}}$$

Exercise: show $[N, a^+_{\bar{q}}] = a^+_{\bar{q}}$.

Show also that

$$N \,|p_1 \ldots p_n\rangle = n \,|p_1 \cdots p_n\rangle$$

and $[N, H] = 0$

Note: $[N, H] = 0$ implies $\dot{N} = 0$

- the number of particles is conserved

This is true for free theories (no interact.)

but not in general!