Wick's theorem

Recall: for $\hat{H} = \hat{H}_0 + \hat{H}_I$, we introduced interact. picture (or inter. represent.), where operators $\hat{O}_{i.p.}(x)$ evolve in time with $\hat{H}_0$:

$$\frac{d}{dt}\hat{O}_{ip}(x) = \frac{i}{\hbar}\left[\hat{H}_0, \hat{O}_{ip}(x)\right]$$

– as in free theory, but the states obey

$$i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle_{ip} = \hat{H}_I^{ip}|\varphi(t)\rangle_{ip}.$$

This eq. can be converted into integral eq. and solved by iterations:

$$|\varphi(t_f)\rangle_{ip} = T\exp\left[\frac{-i}{\hbar}\int_{t_i}^{t_f}\hat{H}_I^{ip}(\tau)d\tau\right]|\varphi(t_i)\rangle_{ip}.$$

(Dyson, 1949).

Scattering problem:

$t_i \to -\infty \quad |\varphi(t_i)\rangle_{ip} \to |i\rangle$

$t_f \to +\infty \quad |\varphi(t_f)\rangle_{ip} \to |f\rangle$

$$|f\rangle = \hat{S}|i\rangle$$ (10)

$$\hat{S} = T\exp\left[-\frac{i}{\hbar}\int d^4x\, \hat{\mathcal{H}}_I^{i.p.}(x)\right]$$

$$S_{fi} = \langle f|\hat{S}|i\rangle \qquad \hat{S}^\dagger\hat{S} = \mathbb{1}\ :$$

Unitary operator (conserv. of prob.).

Convenient to define $\hat{T}$: $\hat{S} = 1 + i\hat{T}$

Exercise: show that unitarity implies

$$2\,\mathrm{Im}\, T_{ii} = \sum_f |T_{if}|^2$$

- Computing S_{fi} is equiv. to computing

$$\langle f|\, T\left(\mathcal{H}_I^{ip}(x_1)\cdots\mathcal{H}_I^{ip}(x_n)\right)|i\rangle$$

e.g. for $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$,

$$\mathcal{H}_I(x) = \frac{\lambda}{4!}\phi^4(x)$$

$$\mathcal{H}_I^{i.p.}(x) = \frac{\lambda}{4!}\phi_{ip}^4(x)\ :\ \text{expand in } a, a^\dagger.$$

- Consider $\langle 0|\, T\phi_{ip}(x)\,\phi_{ip}(y)\,|0\rangle$

$$\phi_{ip}(x) = \phi^+_{ip} + \phi^-_{ip},$$

$$\phi^+_{ip} = \int d^3\tilde{p}\, a_{\bar{p}}\, e^{-ipx} \qquad \phi^-_{ip} = \int d^3\tilde{p}\, a^+_{\bar{p}}\, e^{ipx}$$

By construction, $\phi^+_{ip}|0\rangle = 0$ and

$\langle 0|\phi^-_{ip} = 0.$

- $x^0 > y^0$: $T\phi_{ip}(x)\phi_{ip}(y) = \phi(x)\phi(y) =$

$$= \left(\phi^+_{ip}(x) + \phi^-_{ip}(x)\right)\left(\phi^+_{ip}(y) + \phi^-_{ip}(y)\right) =$$

$$= \phi^+_{ip}(x)\phi^+_{ip}(y) + \phi^-_{ip}(x)\phi^+_{ip}(y) + \phi^-_{ip}(x)\phi^-_{ip}(y)$$

$$+ \phi^-_{ip}(y)\phi^+_{ip}(x) + [\phi^+_{ip}(x), \phi^-_{ip}(y)] =$$

$$= :\phi_{ip}(x)\phi_{ip}(y): + [\phi^+_{ip}(x), \phi^-_{ip}(y)]$$

↑ all terms are $\sim a^+a$.

- $x^0 < y^0$: $T\phi_{ip}(x)\phi_{ip}(y) = \phi_{ip}(y)\phi_{ip}(x) =$

$$= :\phi(x)\phi(y): + [\phi^+_{ip}(y), \phi^-_{ip}(x)].$$

Define the contraction of $\phi(x)$, $\phi(y)$:

$$\wick{\c1\phi_{ip}(x)\c1\phi_{ip}(y)} = \begin{cases} [\phi^{+}_{ip}(x)\,\phi^{-}_{ip}(y)], & x^0 > y^0 \\ [\phi^{+}_{ip}(y)\,\phi^{-}_{ip}(x)], & x^0 < y^0 \end{cases}$$

Observe: $\wick{\c1\phi_{ip}(x)\c1\phi_{ip}(y)} = D_F(x-y)$.

Then:

$$T\left(\phi_{ip}(x)\,\phi_{ip}(y)\right) = :\phi_{ip}(x)\phi_{ip}(y): + \wick{\c1\phi_{ip}(x)\c1\phi_{ip}(y)}.$$

<u>In general</u>:

$$T\left(\phi_{ip}(x_1)\cdots\phi_{ip}(x_n)\right) = :\phi_{ip}(x_1)\cdots\phi_{ip}(x_n): +$$

$$+ :\text{all possible contractions}:$$

(Wick, 1950; Dyson, 1951)

<u>example</u>: $T\left(\phi_{ip}(x_1)\,\phi_{ip}(x_2)\phi_{ip}(x_3)\right) =$

$$= :\phi_{ip}(x_1)\phi_{ip}(x_2)\phi_{ip}(x_3): + :\wick{\c1\phi_{ip}(x_1)\,\c1\phi_{ip}(x_2)}\phi_{ip}(x_3):$$

$$+ :\phi_{ip}(x_1)\,\wick{\c1\phi_{ip}(x_2)\,\c1\phi_{ip}(x_3)}: + :\wick{\c1\phi_{ip}(x_1)\phi_{ip}(x_2)\c1\phi_{ip}(x_3)}:$$

__example:__ $T\left(\phi_{ip}(x_1)\,\phi_{ip}(x_2)\,\phi_{ip}(x_3)\,\phi_{ip}(x_4)\right) =$

$$= :\phi_{ip}(x_1)\,\phi_{ip}(x_2)\,\phi_{ip}(x_3)\,\phi_{ip}(x_4): +$$

$$+ :\wick{\c1\phi_{ip}(x_1)\,\c1\phi_{ip}(x_2)}\,\phi_{ip}(x_3)\,\phi_{ip}(x_4): +$$

$$+ :\wick{\c1\phi(x_1)\,\phi(x_2)\,\c1\phi(x_3)}\,\phi(x_4): +$$

$$+ :\wick{\c1\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\c1\phi(x_4)}: +$$

$$+ :\phi(x_1)\,\wick{\c1\phi(x_2)\,\c1\phi(x_3)}\,\phi(x_4): +$$

$$+ :\phi(x_1)\,\wick{\c1\phi(x_2)\,\phi(x_3)\,\c1\phi(x_4)}: +$$

$$+ :\phi(x_1)\,\phi(x_2)\,\wick{\c1\phi(x_3)\,\c1\phi(x_4)}: +$$

$$+ :\wick{\c1\phi(x_1)\,\c1\phi(x_2)\,\c2\phi(x_3)\,\c2\phi(x_4)}: +$$

$$+ :\wick{\c1\phi(x_1)\,\c2\phi(x_2)\,\c1\phi(x_3)\,\c2\phi(x_4)}: +$$

$$+ :\wick{\c1\phi(x_1)\,\c2\phi(x_2)\,\c2\phi(x_3)\,\c1\phi(x_4)}:$$

Note: $\langle 0|$ any uncontracted normal-ordered $\phi_{ip}(x)\,|0\rangle = 0$

$$\Rightarrow \langle 0|\, T\big(\phi_{ip}(x_1)\,\phi_{ip}(x_2)\big)|0\rangle = D_F(x_1 - x_2)$$

$$\langle 0|\, T\big(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\big)|0\rangle = 0$$

(the same for $\forall$ number of $\phi_{ip}(x)$ which is odd)

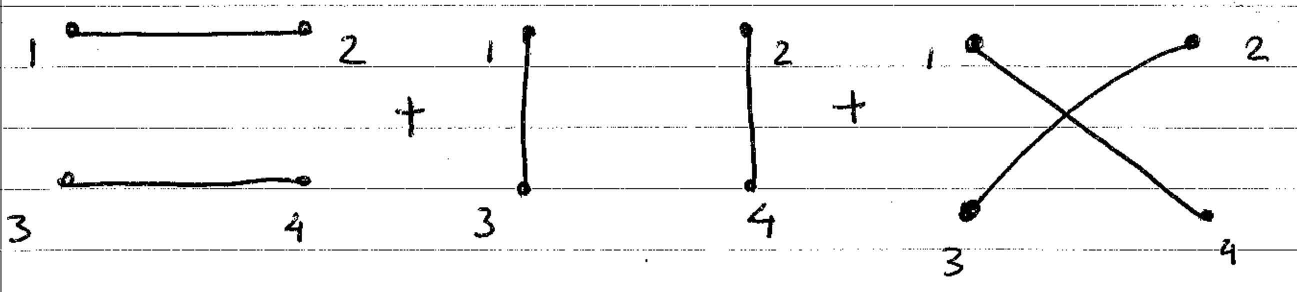
$$\langle 0|\, T\big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\big)\, 0\rangle =$$

$$= D_F(x_1 - x_2)\, D_F(x_3 - x_4) +$$

$$+ D_F(x_1 - x_3)\, D_F(x_2 - x_4) +$$

$$+ D_F(x_1 - x_4)\, D_F(x_2 - x_3) \quad =$$

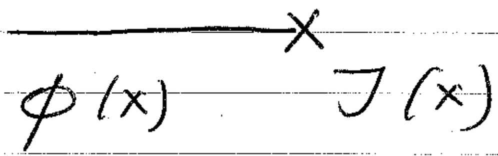
=



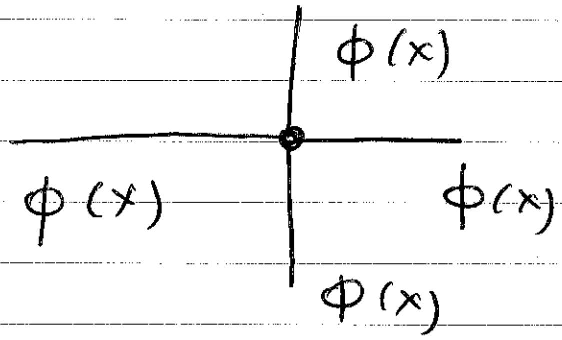
Feynman diagrams.

Examples of $\mathcal{H}_I(x)$

- $\mathcal{H}_I(x) = J(x)\,\phi(x)$ (external source)

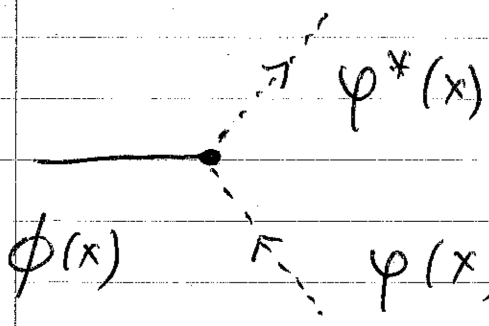


- $\mathcal{H}_I(x) = \frac{\lambda}{4!}\phi^4(x)$ $\left(\sim \phi J,\ J \sim \phi^3\right)$



- $\mathcal{H}_I(x) = g\,\varphi^*(x)\,\varphi(x)\,\phi(x)$

$\left(\sim J\phi,\ J \sim \varphi^*\varphi\right)$



Example: $\mathcal{H}_I^{ip} = \frac{\lambda}{4!}\phi_{ip}^4$

Consider $S_{fi} = \langle f | \hat{S} | i \rangle$, where

$$|i\rangle = |p\rangle = a_{\bar{p}}^{+} |0\rangle$$

$$|f\rangle = |q\rangle = a_{\bar{q}}^{+} |0\rangle$$

$$S_{fi} \equiv \mathcal{A} = \langle q | T \exp\left[\frac{-i}{\hbar}\int d^4x\, \mathcal{H}_I^{ip}(x)\right] | p \rangle$$

$$S = T\Big(\mathbb{1} - \frac{i\lambda}{4!}\int d^4x\, \phi_{ip}^4(x) -$$

$$- \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \int d^4x_1\, d^4x_2\ \phi_{ip}^4(x_1)\phi_{ip}^4(x_2) + \dots \Big)$$

$$\mathcal{A} = \langle 0 | a_{\bar{q}}\, a_{\bar{p}}^{+} | 0 \rangle - \frac{i\lambda}{4!}\int d^4x \langle 0 | T a_{\bar{q}}\, \phi_{ip}^4(x)\, a_{\bar{p}}^{+} | 0 \rangle$$

$$+ O(\lambda^2)$$

- $\langle 0 | a_{\bar{q}}\, a_{\bar{p}}^{+} | 0 \rangle = \langle q | p \rangle = (2\pi)^3 2\omega_{\bar{p}}\, \delta^{(3)}(\bar{p} - \bar{q})$
- $\langle 0 | T a_{\bar{q}}\, \phi_{ip}(x)\phi_{ip}(x)\phi_{ip}(x)\phi_{ip}(x) a_{\bar{p}}^{+} | 0 \rangle$

Wick's theorem $\Rightarrow$ 2 types of contractions

1) $a_q \; \phi \; \phi \; \phi \; \phi \; a_p^+$

$a_q \; \phi \; \phi \; \phi \; \phi \; a_p^+$

$a_q \; \phi \; \phi \; \phi \; \phi \; a_p^+$

They contribute $3 \langle 0 | a_q a_p^+ | 0 \rangle \langle 0 | T \phi \phi | 0 \rangle$

$\times \langle 0 | T \phi \phi | 0 \rangle$

2) $a_q \; \phi \; \phi \; \phi \; \phi \; a_p^+$ and similar ones

$\Rightarrow 12 \langle 0 | a_q \phi_{in}(x) | 0 \rangle \langle 0 | T \phi_{in}(x) \phi_{in}(x) | 0 \rangle$

$\times \langle 0 | \phi_{in}(x) a_p^+ | 0 \rangle$

- $\langle 0 | \phi_{in}(x) a_p^+ | 0 \rangle =$

$= \langle 0 | \int d^3\tilde{q} \left(a_q e^{-iqx} + a_q^+ e^{iqx} \right) a_p^+ | 0 \rangle =$

$= \int d^3\tilde{q} \langle 0 | a_q a_p^+ | 0 \rangle e^{-iqx} = e^{-ipx}.$

Similarly, $\langle 0 | a_q \phi_{in}(x) | 0 \rangle = e^{iqx}$.

Recall also:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{-ik(x-y)}$$

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}$$

We now have all ingredients to compute part 2):

$$\mathcal{A}_1^{(2)} = -\frac{i\lambda}{4!} 12 \int d^4x \, e^{-i(p-q)x} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon}$$

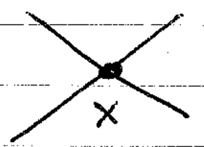
$$= \frac{\lambda}{2} \delta^{(4)}(p-q) \int \frac{d^4k}{k^2-m^2+i\varepsilon}$$

What about part 1)?

But first, we draw Feynman diagrams for $\mathcal{A}_1^{(2)}$!

- draw interaction vertices at $x_1 \dots x_n$

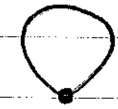
Here we have only one:

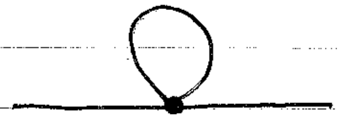


- incoming lines : $\overline{\phi(x)\, a_p^+}$

outgoing lines: $\overline{a_q\, \phi(x)}$

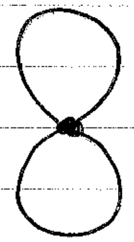
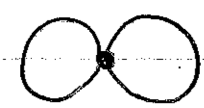
- propagators $D_F(x-y)$ connect vertices

Here $y = x$, so we have 

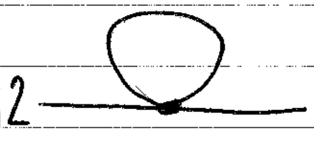
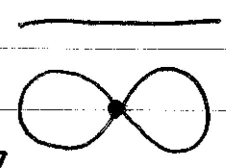
Together : 

Now, part 1) has a piece not related to x : $\langle 0 | a_q\, a_p^+ | 0 \rangle$ represented by

and also x-related piece :

 or 

In total:

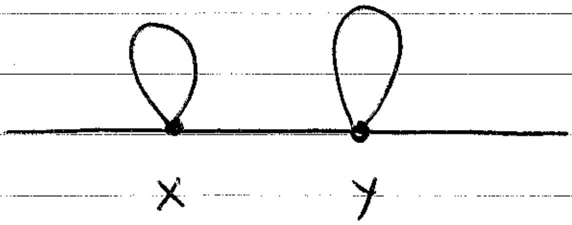
$\mathcal{A} =$  $+ 12$  $+ 3$  $+ O(\lambda^2)$

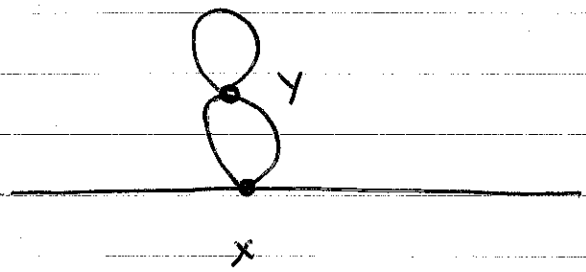
connected / disconnected

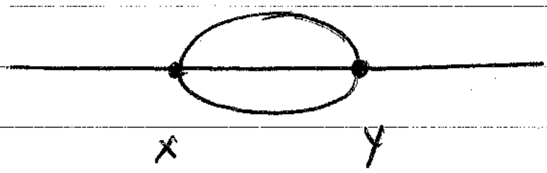
vacuum diagram $\langle 0 | S | 0 \rangle$

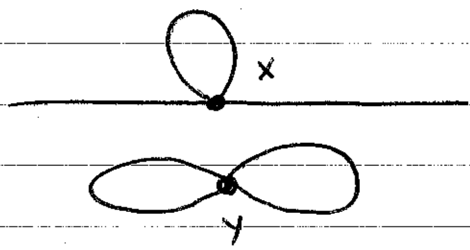
$$\mathcal{A}_2 = -\frac{\lambda^2}{2!(4!)^2} \langle 0 | \iint d^4x\, d^4y\, T[a_q^- \phi_{ip}(x)\phi_{ip}(x)\phi_{ip}(x)\phi_{ip}(x)$$
$$\phi_{ip}(y)\phi_{ip}(y)\phi_{ip}(y)\phi_{ip}(y)\, a_p^+] | 0 \rangle$$

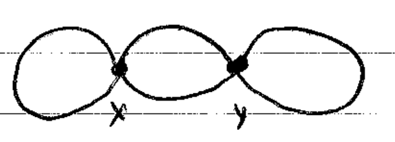
This gives 7 diagrams:

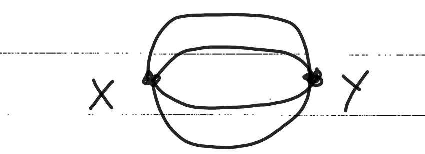


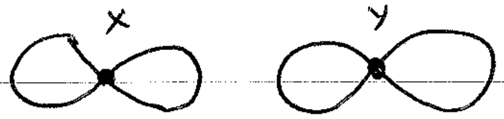






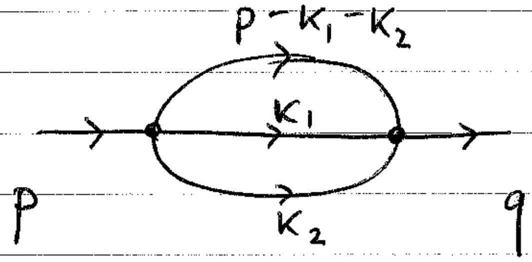






– with appropriate symmetry factors.

Explicitly:



$$\Rightarrow (2\pi)^4 \delta^{(4)}(p-q) \frac{(-i\lambda)^2}{6} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \times$$

$$\times \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{(p-k_1-k_2)^2 - m^2 + i\varepsilon}.$$

Feynman rules for $\lambda\phi^4$ theory
(in momentum space)

- Each vertex gives a factor $-i\lambda$
- Each internal line gives a propagator

$$\frac{i}{k^2 - m^2 + i\varepsilon}$$

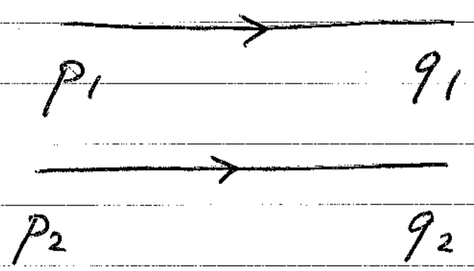
- At each vertex, total momentum is conserved
- Integrate over internal momenta $\int \frac{d^4 k}{(2\pi)^4}$
- Include symmetry factors
- Include overall $(2\pi)^4 \delta^{(4)}(p_f - p_i)$ factor

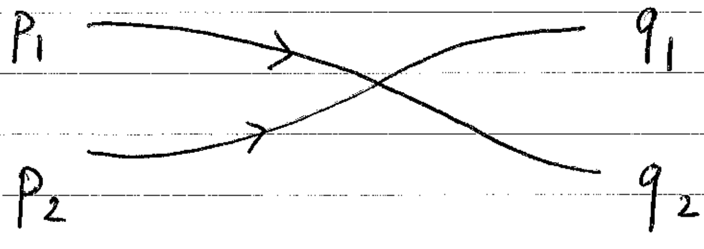
Scattering: $2 \to 2$

Consider now $S_{fi} = \langle q_1 q_2 | \hat{S} | p_2 p_1 \rangle$

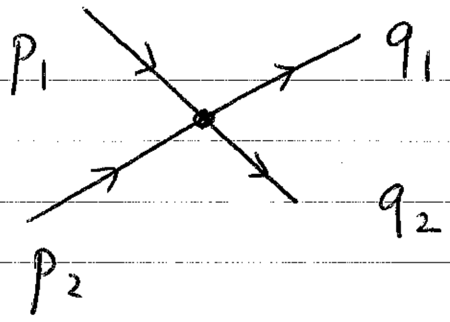
$= \langle 0 | a_{q_1} a_{q_2} \hat{S} a^\dagger_{p_2} a^\dagger_{p_1} | 0 \rangle$

λ^0 :

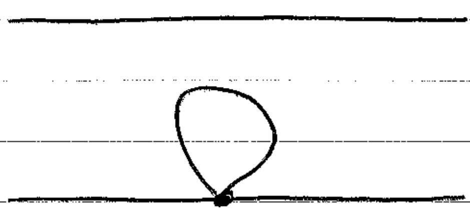


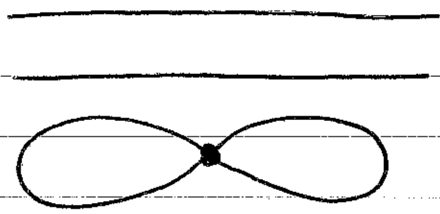


λ :



$(*)$





$+ \; O(\lambda^2)$.

Connected diagram $(*)$

$\Rightarrow (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)(-i\lambda)$

$\Rightarrow$ lowest order trans. prob. $\sim |S_{fi}|^2 \sim \lambda^2$.

In $\lambda\phi^4$ theory, expect scattering cross-section $\sim \lambda^2$, $\lambda \ll 1$.