Canonical quantization

In classical mechanics (Hamiltonian formalism) functions $f(q,p)$ on phase space satisfy e.o.m.

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_a}\dot{q}_a + \frac{\partial f}{\partial p_a}\dot{p}_a$$

With Hamilton's eqs

$$\begin{cases} \dot{q}_a = \partial H / \partial p_a \\ \dot{p}_a = -\partial H / \partial q_a \end{cases}$$

this is

$$\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_a}\frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial H}{\partial q_a} =$$

$$= \frac{\partial f}{\partial t} + \{f, H\}_{PB}$$

For $\forall A, B, C$ on phase space,

$$\{A, B\}_{PB} = -\{B, A\}_{PB}$$

$$\{A, \{BC\}\}_{PB} + \{C, \{A,B\}\}_{PB} + \{B, \{C,A\}\}_{PB} = 0$$

This is Lie algebra of functions on phase space with PB as Lie bracket.

Symmetries = canonical transformat. $(p, q) \to (P, Q)$ preserving PB.

Quantization: map this algebra of functions into algebra of operators on a Hilbert space with

$$[\hat{A}, \hat{B}] = i\hbar \{\hat{A}\, \hat{B}\}_{PB}$$

(Dirac, 1925)

Then $[\hat{q}_a\, \hat{p}_b] = i\hbar\, \delta_{ab}$

$$[\hat{q}_a\, \hat{q}_b] = 0$$

$$[\hat{p}_a\, \hat{p}_b] = 0$$

$$\dot{\hat{q}}_a(t) = i\hbar\, [\hat{H}, \hat{q}_a(t)] \qquad \dot{\hat{p}}_a(t) = i\hbar\, [\hat{H}, \hat{p}_a(t)]$$

- Will drop $\hbar$ and hats from now on

Heisenberg picture: $A(t)$, states in Hilbert space are static.

Note: this map is not 1-1: quantum systems are more general than classical which only appear (uniquely) in the limit „$\hbar \to 0$" i.e. one can „restore" classical system from quantum by taking the limit, but the opposite is not true.

In field theory:

$$[\phi_a(t,\bar{x}), \pi_b(t,\bar{y})] = i\delta^{(3)}(\bar{x}-\bar{y})\delta_{ab}$$

$$[\phi_a(t,\bar{x})\, \phi_b(t,\bar{y})] = 0$$

$$[\pi_a(t,\bar{x})\, \pi_b(t,\bar{y})] = 0$$

$$\begin{cases} \dot{\phi}_a(t,\bar{x}) = i[H, \phi_a(t,\bar{x})] \\ \dot{\pi}_a(t,\bar{x}) = i[H, \pi_a(t,\bar{x})] \end{cases}$$

Quantization of real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad \left(\text{free field}\right)$$

E.o.m. $\left(\Box + m^2\right)\phi = 0$

Classical solution is found via Fourier

$$\phi(t,\bar{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\bar{p}\bar{x}}\, \phi(t,\bar{p})$$

$$\Rightarrow \quad \ddot{\phi}(t,\bar{p}) + \omega_{\bar{p}}^2\, \phi(t,\bar{p}) = 0$$

$$\omega_{\bar{p}} = \sqrt{|\bar{p}|^2 + m^2}\,.$$

$$\phi(t,\bar{p}) = a_{\bar{p}}^{(1)}\, e^{-i\omega_{\bar{p}} t} + a_{\bar{p}}^{(2)}\, e^{i\omega_{\bar{p}} t}$$

Since $\phi(t,\bar{x})$ is real, $a_{\bar{p}}^{(2)} = a_{\bar{p}}^{(1)*}$

We can write $a_{\bar{p}}^{(1)} \equiv N_{\bar{p}}\, a_{\bar{p}}$ with a (real) normalisation $N_{\bar{p}}$ to be chosen later. Then

$$\phi(t,\bar{x}) = \int \frac{d^3p}{(2\pi)^3}\, N_{\bar{p}} \left(a_{\bar{p}}\, e^{-ipx} + a_{\bar{p}}^{*}\, e^{ipx}\right),$$

where $px = \omega_{\bar{p}} t - \bar{p}\bar{x} = p^0 x^0 - \bar{p}\bar{x}$.

$$\pi(t,\bar{x}) = \dot{\phi}(t,\bar{x}) = \int \frac{d^3q}{(2\pi)^3}\, N_{\bar{q}} \left(a_{\bar{q}}\, e^{-iqx} - a_{\bar{q}}^{*}\, e^{iqx}\right) \times (-i\omega_{\bar{q}})\,.$$

We now promote classical fields to (Hermitian) operators and impose

$$[\phi(t,\bar{x})\ \pi(t,\bar{y})] = i\,\delta^{(3)}(\bar{x}-\bar{y})$$

etc.

$$a_{\bar{p}},\ a^{*}_{\bar{p}} \rightarrow \hat{a}_{\bar{p}},\ \hat{a}^{+}_{\bar{p}}$$

<u>Exercise</u>: show that with $[\hat{a}_{\bar{p}}\ \hat{a}^{+}_{\bar{q}}] =$

$$= (2\pi)^3\, 2\omega_{\bar{p}}\, \delta^{(3)}(\bar{p}-\bar{q})$$

$$[\hat{a}_{\bar{p}}, \hat{a}_{\bar{q}}] = 0, \quad [\hat{a}^{+}_{\bar{p}}, \hat{a}^{+}_{\bar{q}}] = 0$$

and $N_{\bar{p}} = 1/2\omega_{\bar{p}}$ the correct commut. rel. for ϕ and π are recovered.

<u>Note</u>: $\delta(\alpha) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{i\alpha\tau}\, d\tau$.

<u>Note</u>: combination $2\omega_{\bar{p}}\,\delta^{(3)}(\bar{p}-\bar{q})$ is Lorentz-invar (see lecture notes).

<u>Now compute the Hamiltonian</u>

Classically, $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\phi\,\partial_i\phi + \frac{1}{2}m^2\phi^2$

$$H = \int d^3x\, \mathcal{H}$$

Exercise: show that

$$H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\bar p}}\,\omega_{\bar p}\left(\hat a_p \hat a_p^+ + \hat a_p^+ \hat a_p\right) =$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2}a_{\bar p}^+ a_{\bar p} + (2\pi)^3 \frac{\omega_p}{2}\,\delta^{(3)}(0)\right)$$

Vacuum energy

Define vac. state by the condition

$$a_{\bar p}\,|0\rangle = 0 \qquad \forall \bar p$$

Then

$$H\,|0\rangle = E_0\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{\omega_p}{2}(2\pi)^3 \delta^{(3)}(0)\,|0\rangle$$

$$(2\pi)^3 \delta^{(3)}(0) = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3\bar x\, e^{i\bar p\bar x}\Big|_{\bar p = 0} =$$

$$= \lim_{L\to\infty} L^3 = V \qquad (\text{3-volume})$$

Define energy density $\varepsilon = E/V$