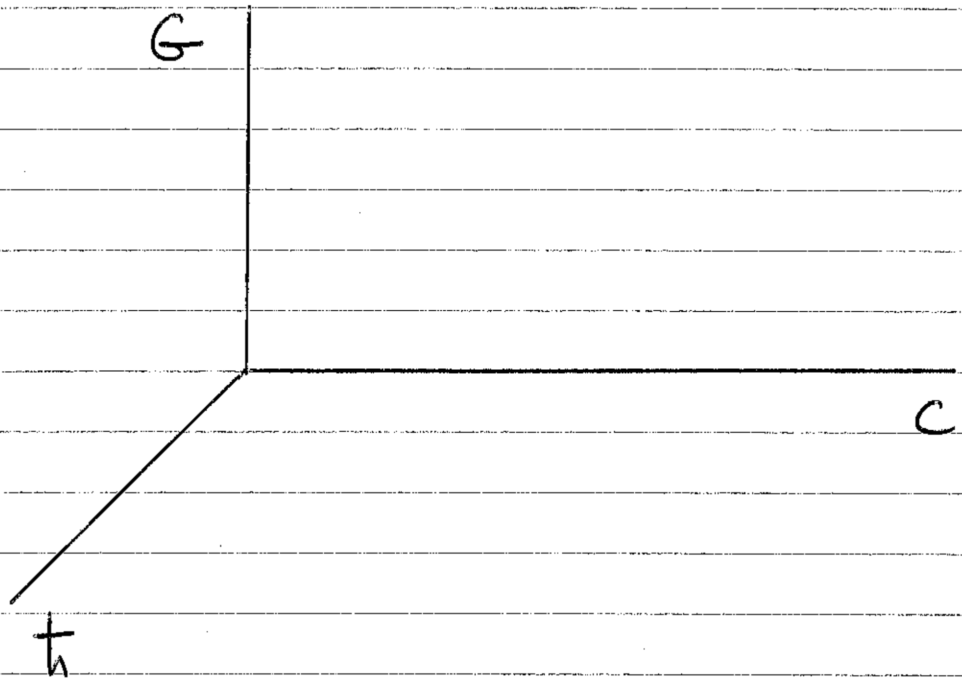
C6: Introduction to Quantum Field Theory

Goal: extrapolate QM to relativistic domain (i.e. beyond $v/c \ll 1$, $E/mc^2 \ll 1$)



Appropriate dimensionless parameters:

SR: v/c

QM: $\lambda/l \qquad \lambda = \dfrac{\hbar}{mc}$

QG: l_p/l

Reminder: Planck units

$$l_p = \sqrt{\frac{G\hbar}{c^3}} \sim 1.6 \cdot 10^{-33}\,\text{cm}$$

$$t_p = \sqrt{\frac{G\hbar}{c^5}} \sim 0.5 \cdot 10^{-43}\, s$$

$$m_p = \sqrt{\frac{\hbar c}{G}} \sim 2 \cdot 10^{-8}\, kg$$

$$E_p \sim 1.2 \cdot 10^{19}\, GeV \quad (E_{LHC} \sim 10^4\, GeV)$$

<u>Note</u>: grav. coupling has dim. of L^2

$$[G] = L^2$$

<u>Note</u>: quantum grav. becomes important for $\lambda \sim l$ (size of grav. system, e.g. $l \sim r_s = 2GM/c^2$)

$$\frac{2MG}{c^2} \sim \frac{\hbar}{Mc}$$

$$\Rightarrow M \sim m_p \quad (\text{Planck mass})$$

<u>Moral</u>: for $E \ll E_p$ can treat gravity as classical background. Moreover, to a good approx., $g_{\mu\nu}(x) \simeq \eta_{\mu\nu}$ (Mink)

$\Rightarrow$ QFT in flat space-time.

4d Mink $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

Symmetries: 4 transl. ($t \to t+a$ etc)

Lorentz transf. $\begin{cases} 3 \text{ rot.} \\ 3 \text{ boosts } (tx, ty, tz) \end{cases}$

$\Rightarrow$ 10-parameter Poincaré group =

= transl. + Lorentz group

More precisely, $\mathbb{R}^{1,3} \rtimes O(1,3)$

(semi-direct product)

Moral: with gravity non-dynamical and flat, fundamental symmetry of the „arena" is Poincaré symmetry.

Thus, we build a theory which has Poincaré-inv. action and covariant e.o.m.

Note: this is not God-given - validity of SR is an experim. fact; keep looking for violations of Lor-invar.

Where do we expect to find rel. corr. to QM?

Spectroscopy of hydrogen atom

QM: $E_{n\ell} = -\dfrac{\alpha^2}{2n^2}mc^2 = -\dfrac{me^4}{2n^2(4\pi\varepsilon_0)^2\hbar^2}$

$(\ell = 0, \dots n-1)$

where $\alpha = \dfrac{e^2}{4\pi\varepsilon_0\hbar c} \approx 1/137$

$\alpha \approx 0.0072973525693$ (PDG 2018)

$\alpha^{-1} \approx 137.035999084$

Electron moves in atom with $v/c \sim 10^{-2}$

$\Rightarrow$ expect rel. corrections to $E_{n\ell}$ of order $v^2/c^2 \sim 10^{-4} \sim O(\alpha^2)$,

i.e. $E_{n\ell} = -\dfrac{\alpha^2}{2n^2}mc^2 + O(\alpha^4)$

- Generalise Schrödinger eq to include rel. effects

Non-rel. particles / waves (de Broglie)

$E = \bar{p}^2/2m$ disp. relation is

reproduced from wave $\psi \sim e^{-i\omega t + i\bar{p}\bar{x}}$

with $E \to i\hbar\, \partial/\partial t$

$$p_i \to -i\hbar \frac{\partial}{\partial x^i}$$

$\Rightarrow$ eq. for ψ is $\quad i\hbar \dfrac{\partial \psi}{\partial t} = \hat{H}\psi,$

$\hat{H} = \hat{p}^2/2m.$

This eq. is parabolic and is not Lor-covariant.

Parab. eqs (diffusion, Navier-Stokes)

$\Rightarrow$ infinite prop. speed.

E.g. $\partial_t \phi = D\nabla^2\phi$ in n dim:

$$\Rightarrow \phi = \frac{1}{(4\pi D t)^{n/2}}\, e^{-\bar{x}^2/4Dt}$$

With $\phi(t=0, \bar{x}) = \delta(\bar{x})$:

at <u>any</u> $t > 0$, the solution is

non-zero at <u>any</u> $|\bar{x}_*|$, no matter how large $\Rightarrow$ signal propagates with infinite speed.

Need hyperbolic eq. such as Maxwell eq:

$$\Box A^\mu = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) A^\mu = 0$$

to respect SR.

- In SR, the disp. rel. for a free particle is $\mathcal{E}^2/c^2 - \bar{p}^2 = m^2c^2$

With $\mathcal{E} \to i\hbar\partial_t$, $\bar{p}_i \to -i\hbar\, \partial/\partial x_i$ we get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m^2c^2}{\hbar^2}\right)\phi(t,\bar{x}) = 0$$

$$\text{or } (\Box + m^2)\,\phi = 0$$

This is Klein-Gordon-Fock eq (1926), originally derived by Schrödinger (1926)

<u>Note</u>: $(\Box - m^2)\phi = 0$ in $-+++$ metric, where $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$.

KG eq. with external electromag. field $A^\mu = (\Phi/c, \bar{A})$:

$$\begin{cases} \bar{p} \to \bar{p} - \frac{e}{c}\bar{A} \\ E \to E - e\Phi \end{cases}$$

(minimal coupling - this gives correct classical eq. of motion e.g. Lorentz force - see LL vol 2)

$$\left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)^2\psi = \left(\frac{\hbar c}{i}\bar{\nabla} - e\bar{A}\right)^2\psi + m^2c^4\psi$$

Hydrogen atom: $\begin{cases} e\Phi = -Ze^2/r \quad (Z=1) \\ \bar{A} = 0 \end{cases}$

Spectrum:

$$E_{nl} = mc^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{l+1/2} - \frac{3}{4}\right) + \ldots\right]$$

fine structure term disagrees with spectroscopy experim.