Two-loop evaluation of large Wilson loops, with overlap fermions, and the b-quark mass shift

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Abstract

We compute, to two loops in pertubation theory, the fermionic contribution to rectangular $R \times T$ Wilson loops, for different values of R and T.

We use the overlap fermionic action. We also employ the clover action, for comparison with existing results in the literature. In the limit $R, T \to \infty$ our results lead to the shift in the b-quark mass. We also evaluate the perturbative static potential as $T \to \infty$.

Introduction

- We compute the perturbative value of **large Wilson loops** up to two loops, using the clover (SW) and overlap fermions.

- Using the perturbative values of Wilson loops of infinite length, we evaluate the shift of the **b-quark mass**.

- Using the perturbative values of Wilson loops of infinite time extent, we evaluate the **static potential**.

- For the case of clover fermions, we also compare our results with established results.

- The calculation of Wilson loops in lattice perturbation theory is useful in a number of ways:

  - The prediction of a strong coupling constant $a_{\overline{MS}}(m_Z)$ from low energy hadronic phenomenology by means of non-perturbative lattice simulations [1–6].

  - It is employed in the context of mean field improvement programmes of the lattice action and operators [7,8].

- In the limit of infinite time separation, $T \to \infty$, Wilson loops give access to the perturbative quark-antiquark potential. Furthermore in the limit of large distances, $R \to \infty$, the self energy of static sources can be obtained from the potential, enabling the calculation of $\overline{m_b}\left(\overline{m_b}\right)$ from the non-perturbative simulations of heavy-light mesons in the static limit [11].

Calculation of Wilson loops

▷The Wilson loop W, around a closed curve C, is the expectation value of the path ordered product of gauge links:

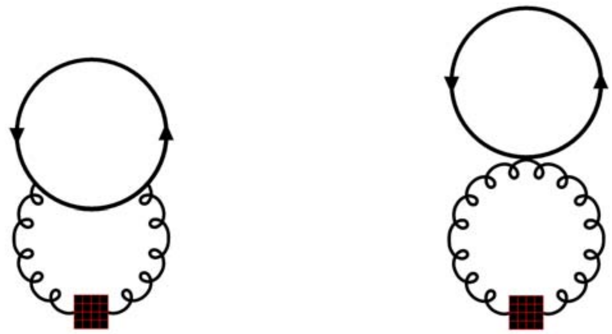
$$W(C) = \left\langle tr \left[\prod_{x_i \in C} U_{x_i,\mu_i} \right] \right\rangle , \tag{1}$$

where $\mu_i \in \pm 1, \cdots, \pm 4$, denotes the direction indicated by $x_{i+1} - x_i$ and $U_{x,-\mu} = U_{x-\hat{\mu},\mu}$. $W(R,T)$, denotes a rectangular Wilson loop where the closed curve contains two opposite lines with an extent of T lattice units pointing in the time direction, separated by a spatial distance aR.

▷ The smallest Wilson loop is the plaquette $\Box$, which is the expectation value of the Wilson gauge action:

$$S_W = \beta \sum_{\Box} \left[1 - \frac{1}{N} \mathrm{Re\ Tr}\left(\Box\right) \right] . \tag{2}$$

▷ To carry out the perturbative procedure we compute the following two Feynman diagrams:



1 **2**

The grid-like square vertex stands for two - point vertex of $W(R,T)$, whose mathematical expression is:

$$\sum_x tr\left(\prod_{x_i\in C} U_{x_i,\mu_i}\right) = -\frac{g^2}{24}\sum_{\mu,\nu}\int \frac{d^4k d^4k'}{(2\pi)^4} A^a_\mu(k)\, A^b_\nu(k') \\ \times\, \delta^{ab}\delta(k+k')\Big[2\delta_{\mu,\nu}\mathrm{S}(k_\mu, R)\sum_\rho \sin^2(k_\rho a T/2) \\ +\, 2\delta_{\mu,\nu}\mathrm{S}(k_\mu, T)\sum_\rho \sin^2(k_\rho a R/2) \\ -\, 4\mathrm{S}(k_\mu, R)\,\mathrm{S}(k_\nu, T)\sin(k_\mu a/2)\sin(k_\nu a/2)\Big], \tag{3}$$

where: $\mathrm{S}(k_\mu, R) \equiv \dfrac{\sin^2(R k_\mu a/2)}{\sin^2(k_\mu a/2)}$.

▷ The involved algebra of the lattice perturbation theory was carried out using our computer package in Mathematica.

▷ The value of each diagram is computed numerically for a sequence of finite lattice sizes. Their values have been summed, and then extrapolated to infinite lattice size.

Calculation with Clover Fermions

▷ The action is, in standard notation:

$$S = S_W + S_f, \tag{4}$$

where:

$$\begin{aligned} S_f = \sum_{x,f} \Big[& (m+4r)\,\overline{\psi}^f_x \psi^f_x - \\ & \frac{1}{2}\sum_{\mu}\left(\overline{\psi}^f_{x+\hat{\mu}}\,(r+\gamma_\mu)\,U^\dagger_\mu(x)\psi^f_x + \overline{\psi}^f_x\,(r-\gamma_\mu)\,U_\mu(x)\psi^f_{x+\hat{\mu}}\right)\Big] \\ & + \frac{i}{4}\,c_{\rm SW}\,\sum_f\sum_{\mu,\nu}\overline{\psi}^f_x\sigma_{\mu\nu}\hat{F}^{\mu\nu}_x\psi^f_x, \end{aligned} \tag{5}$$

and : $\quad \hat{F}^{\mu\nu} \equiv \frac{1}{8}\,(Q_{\mu\nu} - Q_{\nu\mu}), \quad Q_{\mu\nu} = U_{\mu,\nu} + U_{\nu,-\mu} + U_{-\mu,-\nu} + U_{-\nu,\mu}$

Here $U_{\mu,\,\nu}(x)$ is the usual product of link variables $U_\mu(x)$ along a plaquette in the μ-ν directions, originating at x; f is a flavor index; m is the bare fermionic mass; $\sigma_{\mu\nu} = (i/2)[\gamma_\mu,\,\gamma_\nu]$; powers of a may be directly reinserted by dimensional counting. The clover coefficient $c_{\rm SW}$ is a free parameter in the present work; it is normally tuned in a way as to minimize $\mathcal{O}(a)$ effects.

▷The perturbative expansion of the Wilson loop is given by the expression:

$$W(C)\,/\,N = 1 - W_{LO}\, g^2 - \left(W_{NLO} - \frac{(N^2-1)}{N} N_f X\right) g^4, \quad (6)$$

where: W_{LO} and W_{NLO} are the pure gauge contributions with the Wilson gauge action, which can be found in Ref. [9,10], and:

$$X = X_W + X^a_{SW} c_{SW} + X^b_{SW} c^2_{SW}. \quad (7)$$

We have computed the values of X_W, X^a_{SW}, and X^b_{SW}, and we list them in Tables I, II, III.

▷ We compare our results for X_W, X^a_{SW}, and X^b_{SW} with those of F. Di Renzo and L. Scorzato (only X_W) [10]*, and G. S. Bali and P. Boyle [9], for $m = 0$. The comparison can be found in Tables IV, V, VI.

▷Our results for X_W, X^a_{SW}, and X^b_{SW} as a function of mass for square loops, are shown in the graphs below:

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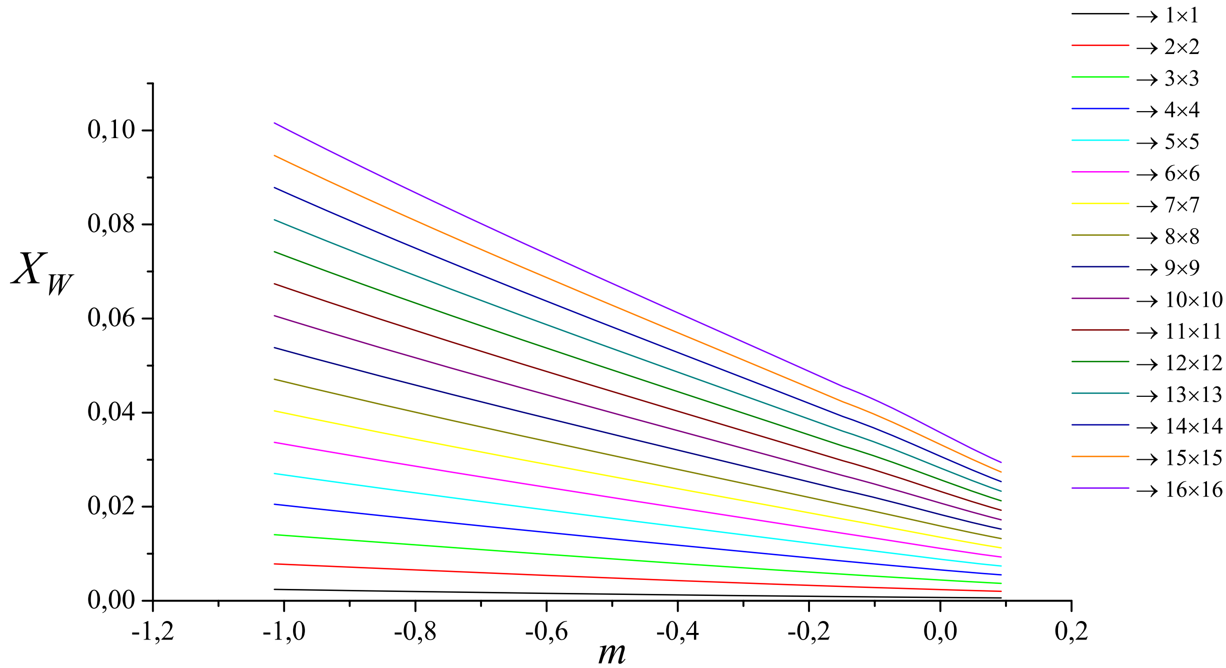
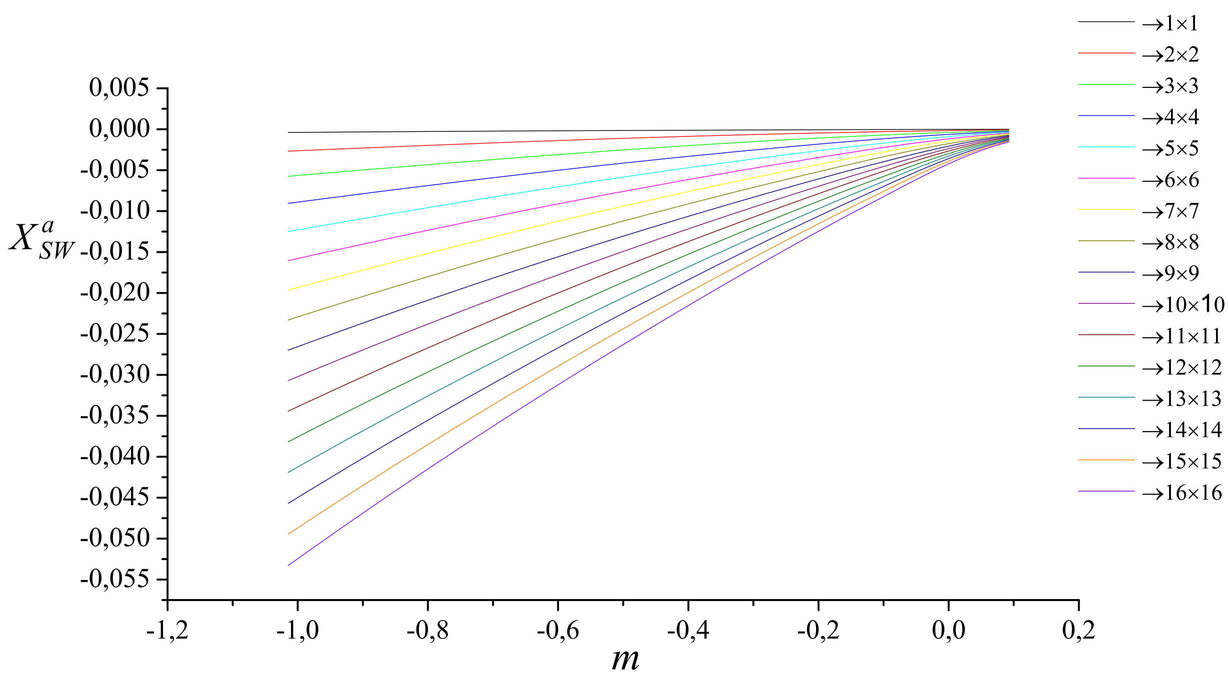
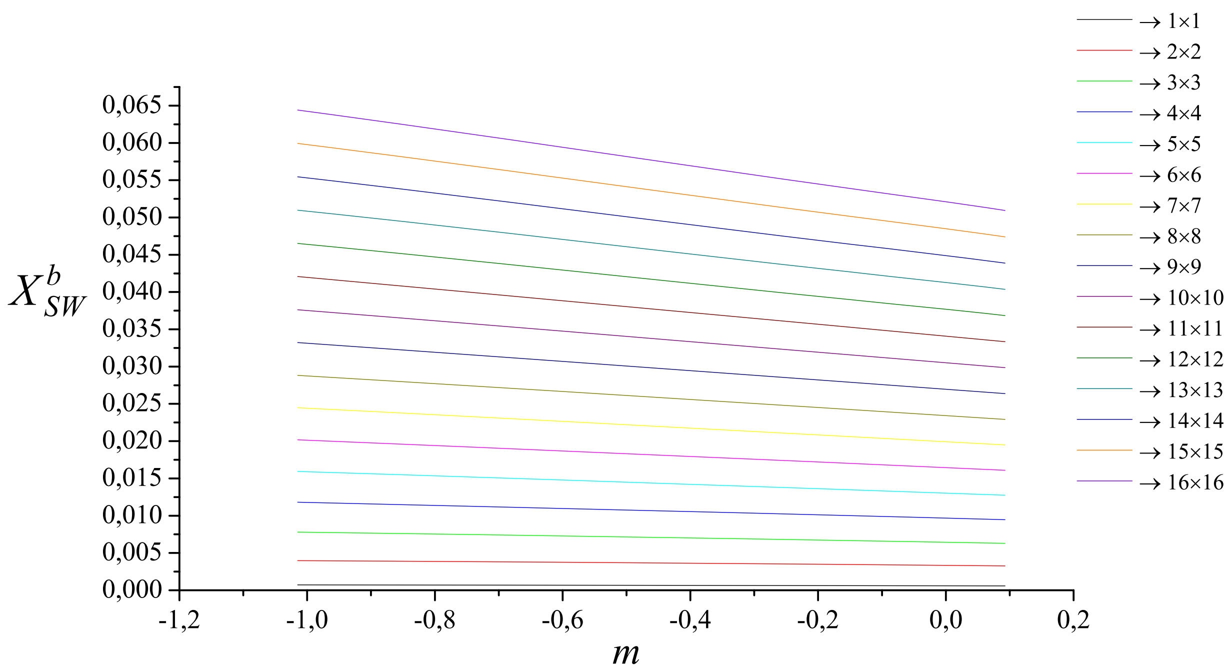
*In order to compare with Ref. [10], we deduce their values of X_W from the data presented there and we estimate the errors stemming from that data.



X_W
m
0,10
0,08
0,06
0,04
0,02
0,00
-1,2
-1,0
-0,8
-0,6
-0,4
-0,2
0,0
0,2
→ 1×1
→ 2×2
→ 3×3
→ 4×4
→ 5×5
→ 6×6
→ 7×7
→ 8×8
→ 9×9
→ 10×10
→ 11×11
→ 12×12
→ 13×13
→ 14×14
→ 15×15
→ 16×16



X^a_{SW}
m
0,005
0,000
-0,005
-0,010
-0,015
-0,020
-0,025
-0,030
-0,035
-0,040
-0,045
-0,050
-0,055
-1,2
-1,0
-0,8
-0,6
-0,4
-0,2
0,0
0,2
→1×1
→2×2
→3×3
→4×4
→5×5
→6×6
→7×7
→8×8
→9×9
→10×10
→11×11
→12×12
→13×13
→14×14
→15×15
→16×16



X^b_{SW}
0,065
0,060
0,055
0,050
0,045
0,040
0,035
0,030
0,025
0,020
0,015
0,010
0,005
0,000
-1,2
-1,0
-0,8
-0,6
-0,4
-0,2
0,0
0,2
m
→ 1×1
→ 2×2
→ 3×3
→ 4×4
→ 5×5
→ 6×6
→ 7×7
→ 8×8
→ 9×9
→ 10×10
→ 11×11
→ 12×12
→ 13×13
→ 14×14
→ 15×15
→ 16×16

Calculation with Overlap Fermions

▷ The fermionic action now reads

$$S_f = \sum_f \sum_{x,y} \bar{\psi}^f_x D_{\rm N}\,(x,y)\,\psi^f_y. \tag{8}$$

with: $D_{\rm N} = M_O\left[1 + X(X^\dagger X)^{-1/2}\right]$, and: $X = D_{\rm W} - M_O$. Here, $D_{\rm W}$ is the massless Wilson-Dirac operator with $r = 1$, and M_O is a free parameter whose value must be in the range $0 < M_O < 2$, in order to guarantee the correct pole structure of $D_{\rm N}$.

▷ Fermionic vertices are obtained by separating the Fourier transform of $D_{\rm N}$ into a free part (inverse propagator D_0) and an interaction part Σ.

$$(1/M_O)D_{\rm N}(q,p) = D_0(p)(2\pi)^4\delta^4(q-p) + \Sigma(q,p), \tag{9}$$

where:

$$D_0^{-1}(p) = \frac{-i\sum_\mu \gamma_\mu \sin p_\mu}{2\left[\omega(p)+b(p)\right]} + \frac{1}{2}, \tag{10}$$

$$\text{and :}\qquad \omega(p) = \left(\sum_\mu \sin^2 p_\mu + [\sum_\mu(1-\cos p_\mu) - M_O]^2\right)^{1/2}, \tag{11}$$

$$b(p) = \sum_\mu(1-\cos p_\mu) - M_O. \tag{12}$$

$\Sigma(q,p)$ is given by:

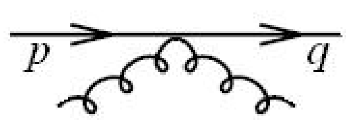
$$\Sigma(q,p) = V^1 + V_1^2 + V_2^2 + \mathcal{O}(g^3), \tag{13}$$

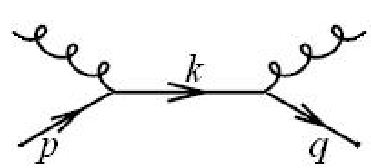
with:

$$\begin{aligned}
V^1 &= \frac{1}{\omega(p)+\omega(q)}\left[X_1(q,p) - \frac{1}{\omega(p)\omega(q)}X_0(q)X_1^\dagger(q,p)X_0(p)\right] \\
V_1^2 &= \frac{1}{\omega(p)+\omega(q)}\left[X_2(q,p) - \frac{1}{\omega(p)\omega(q)}X_0(q)X_2^\dagger(q,p)X_0(p)\right] \\
V_2^2 &= \int \frac{d^4k}{(2\pi)^4}\frac{1}{\omega(p)+\omega(q)}\;\frac{1}{\omega(p)+\omega(k)}\;\frac{1}{\omega(q)+\omega(k)} \\
&\quad \times\Big[-X_0(q)X_1^\dagger(q,k)X_1(k,p) - X_1(q,k)X_0^\dagger(k)X_1(k,p) \\
&\quad - X_1(q,k)X_1^\dagger(k,p)X_0(p) \quad + \frac{\omega(p)+\omega(q)+\omega(k)}{\omega(p)\omega(q)\omega(k)} \\
&\quad \times X_0(q)X_1^\dagger(q,k)X_0(k)X_1^\dagger(k,p)X_0(p)\Big] + \mathcal{O}(g^3).
\end{aligned}$$

X_0, X_1 and X_2 denote the parts of the Dirac-Wilson operator with 0, 1 and 2 gluons (of order $\mathcal{O}(g^0)$,$\mathcal{O}(g^1)$ and $\mathcal{O}(g^2)$).

▷ The 2-gluon vertex splits into two different types of contributions, V_1^2 and V_2^2:



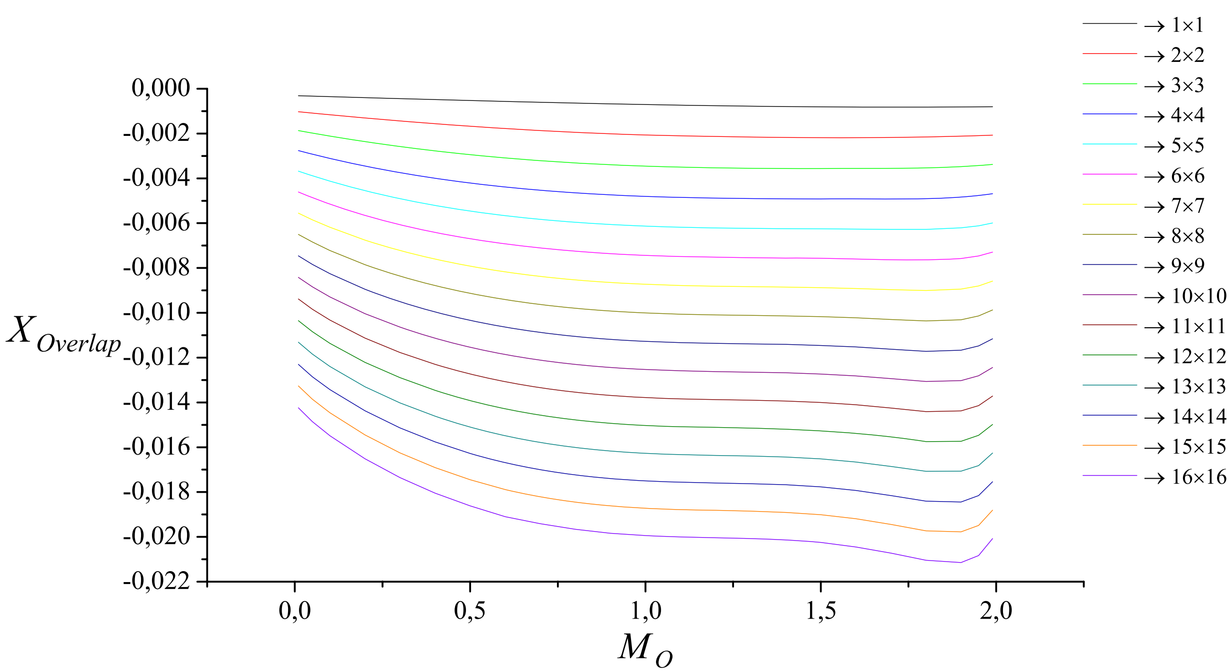


▷ The perturbative expansion of the Wilson loop is given by the expression:

$$W\left(C\right)/N = 1 - W_{LO}g^2 - \left(W_{NLO} - \frac{\left(N^2-1\right)}{N}N_f X_{Overlap}\right)g^4, \tag{14}$$

where $X_{Overlap}$ are the values we compute.

▷ The next graph shows $X_{Overlap}$ as a function of M_O for square Wilson loops. Numerical values are presented in Table VII.



Calculation of the b-quark mass shift

▷In perturbation theory, the expectation value of large Wilson loops decreases exponentially with the perimeter of the loops:

$$\langle W(R,T)\rangle \sim \exp\left(-2\delta m\,(R+T)\right). \tag{15}$$

▷ Following Ref. [11], the perturbative expansion for $\langle W(R,T)\rangle$ is:

$$\langle W(R,T)\rangle = 1 - g^2 W_2(R,T) - g^4 W_4(R,T) + \mathcal{O}\left(g^6\right). \tag{16}$$

▷ Using the expectation value of $W(R,T)$, we obtain the perturbative expansion for δm:

$$\delta m = \frac{1}{2(R+T)}\Big[g^2 W_2(R,T) + g^4 \left(W_4(R,T) + \frac{1}{2}W_2^2(R,T)\right)\Big], \tag{17}$$

where: $W_2(R,T)$ is a pure gluonic contribution and:

$$W_4(R,T) = W_4^g(R,T) + W_4^f(R,T). \tag{18}$$

$W_4^g(R,T)$ is the contribution in the pure-gauge theory and $W_4^f(R,T)$ is the fermionic contribution.

▷ To evaluate the effect of fermions on the mass shift, we must examine their contribution in the limit as $R, T \to \infty$. To this end, we note that our expression assumes the generic form (modulo terms which will not contribute in this limit):

$$\int \frac{d^4p}{(2\pi)^4} \sin^2\left(\frac{p_\nu T}{2}\right) \sin^2\left(\frac{p_\mu R}{2}\right) \left(\frac{1}{\sin^2\left(\frac{p_\nu}{2}\right)} + \frac{1}{\sin^2\left(\frac{p_\mu}{2}\right)}\right) \frac{1}{\left(\hat{p}^2\right)^2} G(p), \quad (19)$$

where $\hat{p}^2 = 4\sum_\rho \sin^2(p_\rho/2)$ and $G(p) \sim p^2$. As $R, T \to \infty$, the above expression becomes:

$$\frac{1}{2}(R+T) \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\left(\hat{\bar{p}}^2\right)^2} G(\bar{p}), \quad (20)$$

where: $\bar{p} = (p_1, p_2, p_3, 0)$ (for $\mu = 4$ or $\nu = 4$).

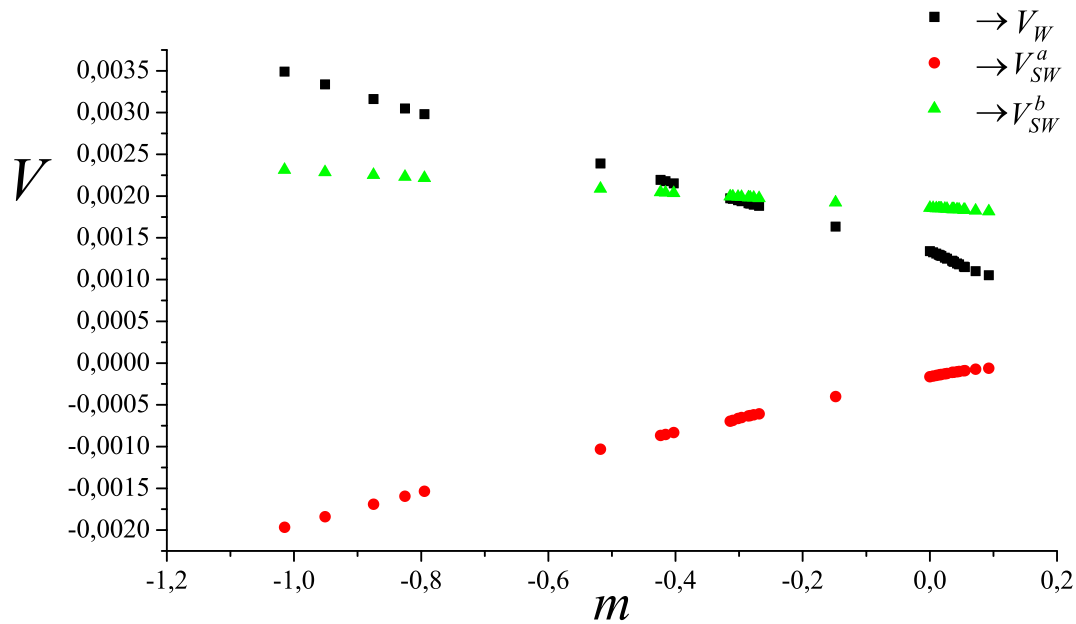
▷ For the clover action, the fermionic contribution takes the form:

$$W_4^f(R,T) = \left(N^2 - 1\right) N_f (R+T) \times \left(V_W + V_{SW}^a c_{SW} + V_{SW}^b c_{SW}^2\right), \quad (21)$$

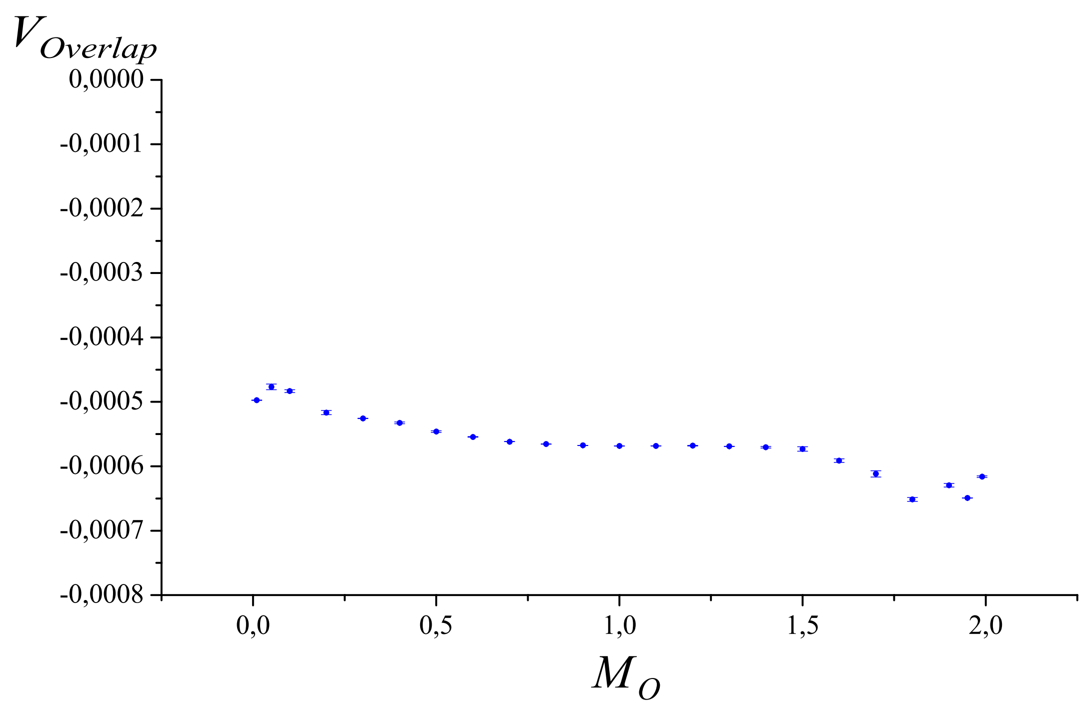
and for the overlap action, it takes the form:

$$W_4^f(R,T) = \left(N^2 - 1\right) N_f (R+T) V_{Overlap}. \quad (22)$$

▷ The values of V_W, V_{SW}^a, V_{SW}^b and $V_{Overlap}$ have been calculated by us. The following two graphs show their values as a function of mass for clover fermions and of M_O for overlap fermions:



$\rightarrow V_W$
$\rightarrow V_{SW}^a$
$\rightarrow V_{SW}^b$
V
0,0035
0,0030
0,0025
0,0020
0,0015
0,0010
0,0005
0,0000
-0,0005
-0,0010
-0,0015
-0,0020
-1,2
-1,0
-0,8
-0,6
-0,4
-0,2
0,0
0,2
m



$V_{Overlap}$
0,0000
-0,0001
-0,0002
-0,0003
-0,0004
-0,0005
-0,0006
-0,0007
-0,0008
0,0
0,5
1,0
1,5
2,0
M_O

▷ At one-loop order the b-quark mass shift is given by:

$$a\delta m \simeq 0.16849 g^2 + \mathcal{O}\left(g^4\right), \tag{23}$$

if we set the number of colours N equal to 3.

Using the results for the infinite time and distance spacing we arrive at the **two-loop** expression for δm. We list below some examples:

▷ The general form of δm for clover fermions is ($\alpha_0 = g^2/4\pi$):

$$a\delta m \simeq 2.1173\alpha_0 + \Big[11.152 - \frac{(4\pi)^2\,(N^2-1)\,N_f}{2N} \times \left(V_W + V^a_{SW} c_{SW} + V^b_{SW} c^2_{SW}\right)\Big]\alpha_0^2 + \mathcal{O}\left(\alpha_0^3\right). \tag{24}$$

The values of $V_W(m)$, $V^a_{SW}(m)$, $V^b_{SW}(m)$ are listed in Table VIII. In particular, setting $m = 0.0$:

$$a\delta m \simeq 2.1173\alpha_0 + \Big[11.152 - \frac{(4\pi)^2\,(N^2-1)\,N_f}{2N}\Big(0.00134096(5) - 0.0001641(1)\,c_{SW} + 0.00185871(2)\,c^2_{SW}\Big)\Big]\alpha_0^2 + \mathcal{O}\left(\alpha_0^3\right). \tag{25}$$

These numbers agree with Ref. [11], within the precision presented there.

▷ For overlap fermions, the general form of δm is:

$$a\delta m \simeq 2.1173\alpha_0 + \Big[11.152 - \frac{(4\pi)^2 \left(N^2 - 1\right) N_f}{2N} V_{Overlap}\Big]\alpha_0^2 + \mathcal{O}\left(\alpha_0^3\right). \quad (26)$$

The values of $V_{Overlap}$, are listed in Table IX. In particular, setting $M_O = 1.4$:

$$a\delta m \simeq 2.1173\alpha_0 + \Big[11.152 - \frac{(4\pi)^2 \left(N^2 - 1\right) N_f}{2N} 0.000571(1)\Big]\alpha_0^2 + \mathcal{O}\left(\alpha_0^3\right). \quad (27)$$

Calculation of the Perturbative Static Potential

▷ The static potential is given by the expression:

$$aV(Ra) = -\lim_{T\to\infty} \frac{d\ln W(R,T)}{dT} \tag{28}$$

$$= V_1(R)\,g^2 + V_2(R)\,g^4 + \ldots, \tag{29}$$

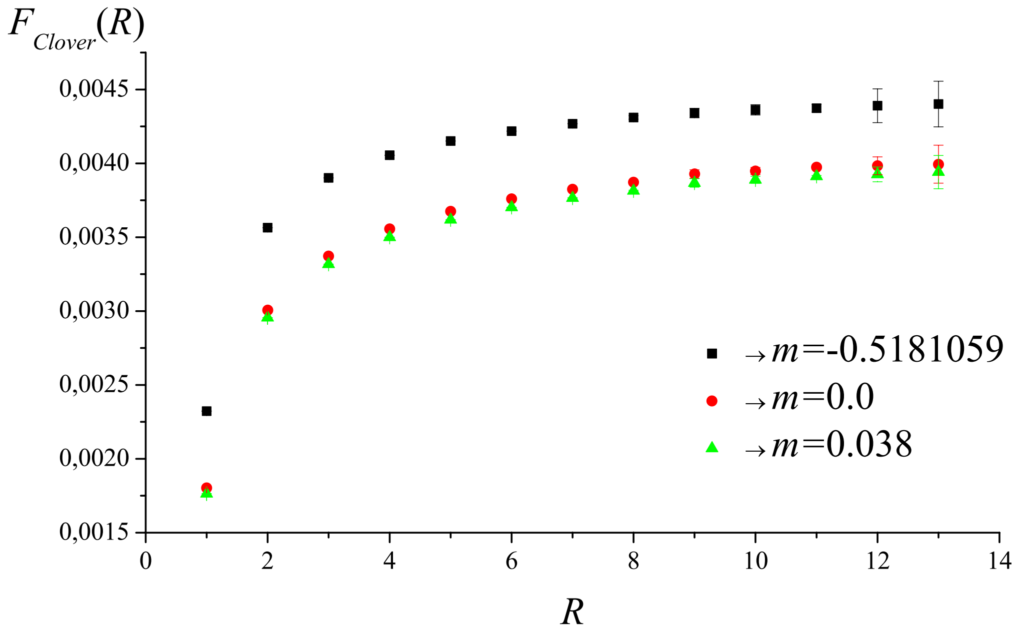
where: $V_1(R)$ is a pure gluonic contribution [9]. $V_2(R)$ contains a gluonic part $V_g(R)$ which can be found in Ref. [9] and a fermionic part $F(R)$:

$$V_2(R) = V_g(R) - \frac{(N^2-1)}{N} N_f F(R). \tag{30}$$

▷ We compute $F(R)$ for clover and overlap fermions. For clover:

$$F(R) = F_{Clover}(R) = F_W(R) + F^a_{SW}(R)\,c_{SW} + F^b_{SW}(R)\,c^2_{SW} \tag{31}$$

and for overlap: $F(R) = F_{Overlap}(R)$. The values of $F_W(R)$, $F^a_{SW}(R)$, $F^b_{SW}(R)$ and $F_{Overlap}(R)$ for specific mass values, can be found in Tables X and XI. We also present them in the following two graphs as a function of R (for $c_{SW} = 1.3$).


$F_{Clover}(R)$
0,0045
0,0040
0,0035
0,0030
0,0025
0,0020
0,0015
0
2
4
6
8
10
12
14
R
→m=-0.5181059
→m=0.0
→m=0.038

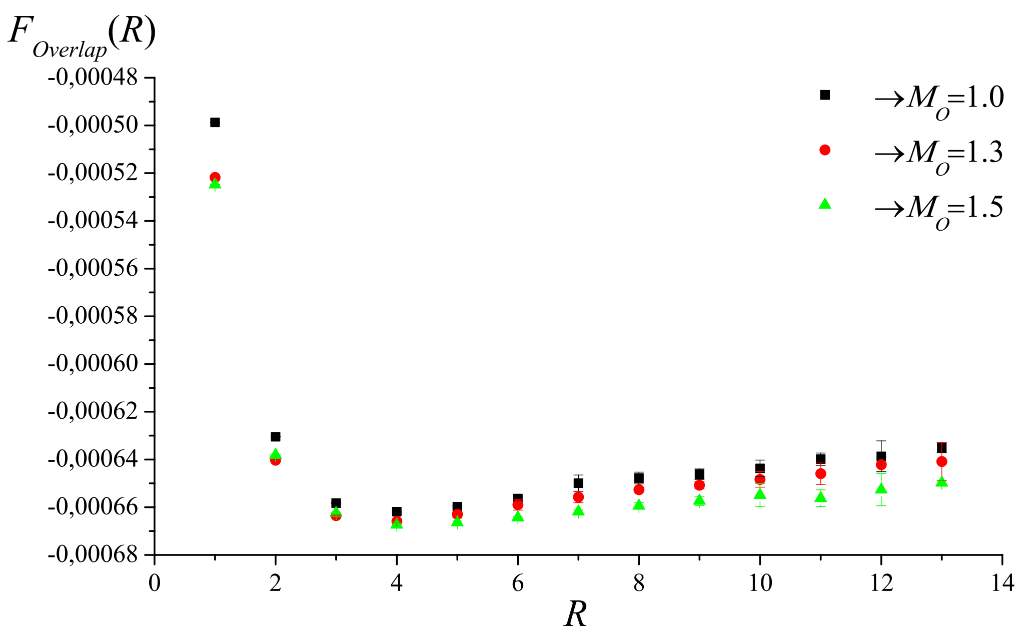

$F_{Overlap}(R)$
-0,00048
-0,00050
-0,00052
-0,00054
-0,00056
-0,00058
-0,00060
-0,00062
-0,00064
-0,00066
-0,00068
0
2
4
6
8
10
12
14
R
→M_O=1.0
→M_O=1.3
→M_O=1.5

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TABLE I. Values of X_W, Eqs.(6, 7), for specific masses.

$R \times T$	-0.8253968	-0.5181059	-0.1482168	0.0	0.038
1×1	0.002017985941(3)	0.0014485866(6)	0.000877986(6)	0.0006929202(2)	0.000651020(4)
2×1	0.003808789532(5)	0.0027649262(6)	0.001680527(8)	0.0013187555(4)	0.001236379(7)
2×2	0.006695252630(2)	0.0049398240(9)	0.00302703(1)	0.0023609937(7)	0.00220774(2)
3×1	0.0054740966(1)	0.0039933034(4)	0.002433213(9)	0.0019053135(4)	0.001784665(8)
3×2	0.0091696433(5)	0.006816343(1)	0.00420265(3)	0.003270793(4)	0.00305476(4)
3×3	0.012099698(1)	0.009076160(3)	0.0056468(1)	0.004385778(7)	0.00409027(4)
4×1	0.0071034996(5)	0.0051941492(2)	0.003169672(9)	0.0024795565(8)	0.00232145(2)
4×2	0.011522152(2)	0.008597002(3)	0.00532068(8)	0.0041372190(9)	0.00386148(4)
4×3	0.014795778(3)	0.011149549(3)	0.0069774(2)	0.005415529(4)	0.00504665(5)
4×4	0.017710525(6)	0.013413919(4)	0.0084565(3)	0.006560910(5)	0.0061084(2)
5×1	0.008722165(1)	0.006386130(1)	0.00390045(1)	0.003049571(1)	0.00285433(3)
5×2	0.013836216(4)	0.010345426(2)	0.0064176(2)	0.004988015(3)	0.00465378(6)
5×3	0.017414883(7)	0.013157873(3)	0.0082648(3)	0.006413278(5)	0.0059734(2)
5×4	0.02050333(1)	0.015574716(3)	0.0098662(4)	0.007654746(6)	0.0071224(3)
5×5	0.02342108(2)	0.017846119(6)	0.0113705(9)	0.00882313(1)	0.0082041(7)
6×1	0.010336971(2)	0.0075747735(4)	0.00462888(3)	0.003617837(3)	0.00338559(4)
6×2	0.016135852(8)	0.0120812913(6)	0.0075056(2)	0.005832123(6)	0.00543991(5)
6×3	0.02000402(2)	0.0151400270(8)	0.0095331(3)	0.007396912(8)	0.0068870(2)
6×4	0.02324722(2)	0.017692706(1)	0.0112452(9)	0.008725234(9)	0.0081151(3)
6×5	0.02626871(3)	0.020056063(2)	0.012828(1)	0.00995767(2)	0.0092550(4)
6×6	0.02919736(5)	0.02233797(1)	0.014349(1)	0.01114489(4)	0.0103532(6)
7×1	0.011950092(5)	0.0087619313(1)	0.00535615(5)	0.004185240(5)	0.00391607(5)
7×2	0.01842902(2)	0.013811440(5)	0.0085891(3)	0.006672898(9)	0.0062228(1)
7×3	0.02257938(3)	0.01710999(1)	0.0107918(3)	0.00837337(2)	0.0077942(3)
7×4	0.02596811(5)	0.01979032(2)	0.0126090(6)	0.00978362(2)	0.0090969(5)
7×5	0.02908270(7)	0.02223621(3)	0.0142608(8)	0.01107437(3)	0.0102893(7)
7×6	0.0320807(1)	0.02457977(4)	0.015838(1)	0.01230793(4)	0.0114291(8)
7×7	0.0350212(2)	0.02687227(8)	0.017374(2)	0.01351051(4)	0.0125406(8)
8×1	0.01356236(1)	0.009948329(4)	0.00608280(8)	0.00475217(1)	0.00444608(5)
8×2	0.02071885(4)	0.01553863(2)	0.0096702(1)	0.00751181(4)	0.0070043(2)
8×3	0.02514752(8)	0.01907354(5)	0.0120453(2)	0.00934576(4)	0.0086978(5)
8×4	0.0286767(1)	0.02187717(8)	0.0139600(6)	0.01083507(7)	0.0100722(7)
8×5	0.0318785(2)	0.0244002(1)	0.015681(1)	0.01218072(7)	0.0113140(8)
8×6	0.0349394(2)	0.0267996(1)	0.017309(1)	0.01345678(7)	0.0124921(8)
8×7	0.0379297(3)	0.0291364(2)	0.018886(1)	0.01469473(8)	0.0136356(9)
8×8	0.0408807(6)	0.0314384(4)	0.020433(1)	0.0159099(2)	0.014758(2)

TABLE II. Values of X^a_{SW}, Eqs.(6, 7), for specific masses.

$R \times T$	-0.8253968	-0.5181059	-0.1482168	0.0	0.038
1×1	-0.000292493808(2)	-0.0001573984(1)	-0.00004774(1)	-0.00002010061(1)	-0.000014684(1)
2×1	-0.000789014304(3)	-0.0004331752(1)	-0.00013418(1)	-0.0000570082(3)	-0.000041862(3)
2×2	-0.00204528413(5)	-0.0011551861(2)	-0.00037017(3)	-0.0001607273(9)	-0.000119471(9)
3×1	-0.001229832201(3)	-0.0006828156(2)	-0.00021266(2)	-0.0000890331(1)	-0.000064740(3)
3×2	-0.0030768902(1)	-0.0017689377(8)	-0.00057553(4)	-0.0002481672(1)	-0.00018354(2)
3×3	-0.0044714337(3)	-0.002630731(5)	-0.00087552(5)	-0.0003758198(2)	-0.00027679(2)
4×1	-0.001650286801(4)	-0.0009202841(3)	-0.00028722(2)	-0.0001190532(1)	-0.000085986(7)
4×2	-0.0040293806(2)	-0.002335182(4)	-0.00076609(5)	-0.0003285984(3)	-0.00024206(2)
4×3	-0.0057054610(5)	-0.003394523(5)	-0.00114482(6)	-0.0004893900(3)	-0.00035909(2)
4×4	-0.007128162(1)	-0.004297068(7)	-0.00147396(7)	-0.0006277562(3)	-0.00045880(6)
5×1	-0.0020638727(1)	-0.0011528916(8)	-0.00035998(2)	-0.0001481833(3)	-0.000106510(8)
5×2	-0.0049550067(3)	-0.002882530(4)	-0.00094971(5)	-0.0004057803(5)	-0.00029804(1)
5×3	-0.0068832569(7)	-0.004118532(9)	-0.00139950(7)	-0.0005962550(5)	-0.00043618(5)
5×4	-0.0084593585(7)	-0.00513455(1)	-0.00177899(7)	-0.0007552273(5)	-0.00055013(8)
5×5	-0.0099050827(9)	-0.00605756(2)	-0.00212377(6)	-0.0008986526(5)	-0.0006522(1)
6×1	-0.0024746461(6)	-0.001383364(1)	-0.00043184(2)	-0.0001768954(5)	-0.000126713(9)
6×2	-0.005869585(2)	-0.003421541(5)	-0.00112989(5)	-0.0004813887(8)	-0.00035280(4)
6×3	-0.008037374(3)	-0.00482471(1)	-0.00164679(6)	-0.000699807(1)	-0.00051074(7)
6×4	-0.009751051(4)	-0.00594233(2)	-0.00207166(6)	-0.000877229(4)	-0.0006373(1)
6×5	-0.011293280(5)	-0.00693732(2)	-0.00245048(4)	-0.001034160(4)	-0.0007484(1)
6×6	-0.012758475(5)	-0.0078745(3)	-0.00280503(1)	-0.001180528(5)	-0.0008517(1)
7×1	-0.002884050(2)	-0.001612786(1)	-0.00050323(2)	-0.0002053982(6)	-0.00014676(1)
7×2	-0.006778774(7)	-0.003956439(1)	-0.00130823(4)	-0.000556193(2)	-0.00040694(4)
7×3	-0.00917979(1)	-0.005521988(2)	-0.00189010(4)	-0.000801642(3)	-0.00058404(8)
7×4	-0.01102290(2)	-0.006735038(6)	-0.00235759(3)	-0.000996348(4)	-0.00072237(9)
7×5	-0.01265212(2)	-0.00779486(1)	-0.002767215(5)	-0.001165438(6)	-0.0008416(2)
7×6	-0.01418382(2)	-0.00878145(2)	-0.00314602(5)	-0.001321187(7)	-0.0009508(4)
7×7	-0.01566445(4)	-0.00972947(2)	-0.0035074(1)	-0.00146965(1)	-0.0010548(7)
8×1	-0.003292706(6)	-0.001841639(5)	-0.00057435(2)	-0.000233789(3)	-0.00016671(2)
8×2	-0.00768502(2)	-0.004489103(6)	-0.00148551(2)	-0.000630558(4)	-0.00046077(4)
8×3	-0.01031575(5)	-0.00621438(2)	-0.002131111(1)	-0.000902522(5)	-0.00065662(4)
8×4	-0.01228369(8)	-0.00751941(3)	-0.00263960(4)	-0.00111385(1)	-0.0008065(1)
8×5	-0.0139944(1)	-0.00863994(3)	-0.0030781(1)	-0.00129432(3)	-0.0009331(1)
8×6	-0.0155865(1)	-0.00967139(3)	-0.0034790(2)	-0.00145857(2)	-0.0010476(2)
8×7	-0.0171157(2)	-0.01065545(6)	-0.0038584(3)	-0.00161390(3)	-0.0011559(2)
8×8	-0.0186086(3)	-0.0116122(2)	-0.0042254(6)	-0.00176401(6)	-0.0012604(3)

TABLE III. Values of X^b_{SW}, Eqs.(6, 7), for specific masses.

$R \times T$	-0.8253968	-0.5181059	-0.1482168	0.0	0.038
1×1	0.000698193815(2)	0.00066565598(1)	0.0006181185(9)	0.00059630769(1)	0.0005903340(3)
2×1	0.00172584505(3)	0.00164432952(2)	0.001528663(3)	0.00147595111(2)	0.0014614713(8)
2×2	0.0038829733(2)	0.00370311435(3)	0.003455127(7)	0.00334255825(8)	0.003311403(2)
3×1	0.0026447690(1)	0.00251041847(3)	0.002327753(4)	0.00224630335(6)	0.0022240270(9)
3×2	0.0055686156(5)	0.0052917398(4)	0.00492817(2)	0.0047675835(2)	0.004723331(3)
3×3	0.007577888(1)	0.0071719960(7)	0.00666547(3)	0.0064495931(7)	0.006390427(3)
4×1	0.0035499739(3)	0.00336406864(7)	0.003114878(7)	0.0030047884(1)	0.002974759(4)
4×2	0.007181537(1)	0.0068154069(4)	0.00634099(3)	0.0061338378(5)	0.006076892(4)
4×3	0.009432576(3)	0.008914872(2)	0.00827670(4)	0.008008705(2)	0.007935550(4)
4×4	0.011436645(5)	0.010795193(5)	0.01001199(4)	0.009688495(3)	0.009600620(4)
5×1	0.0044532013(4)	0.00421590197(7)	0.00390024(2)	0.0037614497(2)	0.003723650(5)
5×2	0.008781297(2)	0.0083272219(9)	0.00774289(6)	0.007489215(1)	0.00741959(2)
5×3	0.011255252(4)	0.010628956(4)	0.00986167(6)	0.009541876(2)	0.00945480(2)
5×4	0.013384587(9)	0.012624976(8)	0.01170146(6)	0.011323119(5)	0.01122068(2)
5×5	0.01543030(1)	0.01454560(1)	0.01347306(7)	0.013037276(8)	0.01291972(5)
6×1	0.0053554122(5)	0.00506673408(7)	0.00468462(3)	0.0045171518(3)	0.004471594(5)
6×2	0.010375729(2)	0.0098339468(8)	0.00914016(4)	0.0088399114(7)	0.00875761(2)
6×3	0.013064833(5)	0.012330656(4)	0.01143529(5)	0.011063834(2)	0.01096286(2)
6×4	0.01530878(1)	0.014432416(8)	0.01337056(8)	0.012937683(5)	0.01282072(6)
6×5	0.01743948(1)	0.01643195(1)	0.01521349(9)	0.014720827(7)	0.01458826(9)
6×6	0.01951938(1)	0.01838393(1)	0.01701323(9)	0.016461641(8)	0.0163137(1)
7×1	0.006256861(1)	0.0059168274(1)	0.00546839(4)	0.0052721785(4)	0.005218838(8)
7×2	0.011966841(2)	0.0113374872(9)	0.01053447(5)	0.0101877335(4)	0.01009275(2)
7×3	0.014866656(3)	0.014024969(1)	0.01300215(6)	0.012579175(2)	0.01246432(5)
7×4	0.017219108(3)	0.016226710(2)	0.01502766(7)	0.014540534(4)	0.01440911(7)
7×5	0.019427331(3)	0.018298159(3)	0.01693557(7)	0.016386498(4)	0.0162390(2)
7×6	0.02156949(1)	0.02030783(1)	0.01878745(8)	0.018177603(4)	0.0180143(2)
7×7	0.02367264(5)	0.02228083(1)	0.0206060(1)	0.01993630(3)	0.0197574(2)
8×1	0.007157780(5)	0.006766415(2)	0.00625165(5)	0.006026753(2)	0.005965640(8)
8×2	0.013555778(5)	0.012838962(4)	0.01192690(5)	0.011533716(8)	0.01142607(1)
8×3	0.016663537(6)	0.015714608(9)	0.01456474(6)	0.014090370(8)	0.01396167(2)
8×4	0.019120705(7)	0.01801277(1)	0.01667727(9)	0.01613611(1)	0.0159903(8)
8×5	0.02140179(1)	0.02015178(5)	0.0186462(1)	0.01804107(2)	0.0178788(8)
8×6	0.02360086(4)	0.02221416(6)	0.0205457(1)	0.01987809(2)	0.0196997(9)
8×7	0.02575120(5)	0.02423078(6)	0.0224037(1)	0.02167474(2)	0.021480(1)
8×8	0.02787067(6)	0.02621858(7)	0.0242355(1)	0.0234460(1)	0.023235(1)

TABLE IV. Here we compare our results for X_W with those of F. Di Renzo and L. Scorzato [10], and G. S. Bali and P. Boyle [9], for $m = 0$.

$R \times T$	Our results	Ref.[10]	Ref.[9]
1×1	0.0006929202(2)	0.000688(3)	0.000696(2)
2×1	0.0013187555(4)	0.001310(8)	0.001326(3)
2×2	0.0023609937(7)	0.00235(2)	0.00238(1)
3×3	0.004385778(7)	0.00431(8)	-
4×4	0.006560910(5)	0.00615(19)	-
5×5	0.00882313(1)	0.00813(42)	-
6×6	0.01114489(4)	0.01010(71)	-
7×7	0.01351051(4)	0.0120(11)	-
8×8	0.0159099(2)	0.0133(18)	-

TABLE V. Here we compare our results for X^a_{SW} with those of G. S. Bali and P. Boyle [9], for $m = 0$.

$R \times T$	Our results	Ref.[9]
1×1	-0.00002010061(1)	-0.0000202(3)
2×1	-0.0000570082(3)	-0.0000565(5)
2×2	-0.0001607273(9)	-0.000161(1)

TABLE VI. Here we compare our results of X^b_{SW} with those of G. S. Bali and P. Boyle [9], for $m = 0$.

$R \times T$	Our results	Ref.[9]
1×1	0.00059630769(1)	0.00059635(1)
2×1	0.00147595111(2)	0.0014759(3)
2×2	0.00334255825(8)	0.0033420(5)

TABLE VII. Values of $X_{Overlap}$, Eq.(14), for specific M_O values.

$R \times T$	$M_O = 0.2$	$M_O = 0.6$	$M_O = 1.0$	$M_O = 1.4$	$M_O = 1.8$
1×1	-0.00039465(1)	-0.00056736986(6)	-0.00070755250(4)	-0.00079691506(6)	-0.0008216755(4)
2×1	-0.00074225(3)	-0.0010460401(2)	-0.0012643519(2)	-0.001379382(1)	-0.001388660(4)
2×2	-0.00130320(3)	-0.0017788121(6)	-0.0020631095(7)	-0.002175237(1)	-0.00215719(3)
3×1	-0.00106622(3)	-0.0014872672(5)	-0.0017747956(4)	-0.001914135(2)	-0.00191097(1)
3×2	-0.00178572(3)	-0.0023943560(8)	-0.002728920(2)	-0.002844382(2)	-0.00281328(8)
3×3	-0.0023584(2)	-0.003088689(3)	-0.003451232(3)	-0.003559630(2)	-0.00352843(9)
4×1	-0.00138312(3)	-0.0019189679(6)	-0.0022756915(8)	-0.002440160(3)	-0.00242470(3)
4×2	-0.0022438(1)	-0.002979068(2)	-0.003366361(3)	-0.003489302(6)	-0.00344688(7)
4×3	-0.0028839(5)	-0.003725538(3)	-0.004122705(5)	-0.004232257(7)	-0.0042034(2)
4×4	-0.0034499(5)	-0.004385664(7)	-0.004804389(5)	-0.00491307(2)	-0.0049016(3)
5×1	-0.00169775(3)	-0.0023482303(6)	-0.002774573(1)	-0.002964467(3)	-0.00293665(5)
5×2	-0.0026941(4)	-0.003555500(3)	-0.003997457(4)	-0.004129009(3)	-0.0040746(2)
5×3	-0.0033921(5)	-0.004346425(4)	-0.004782379(4)	-0.00489521(2)	-0.0048676(2)
5×4	-0.0039898(6)	-0.005021256(4)	-0.005468337(6)	-0.00557929(3)	-0.0055829(2)
5×5	-0.0045506(7)	-0.00566255(1)	-0.006130452(9)	-0.00624369(6)	-0.0062747(2)
6×1	-0.00201153(7)	-0.002776797(1)	-0.003272973(2)	-0.003488349(3)	-0.00344799(8)
6×2	-0.0031406(4)	-0.004129465(2)	-0.004626948(2)	-0.004767304(3)	-0.0047001(2)
6×3	-0.0038940(4)	-0.004962410(3)	-0.005438978(3)	-0.00555545(2)	-0.0055275(3)
6×4	-0.0045189(4)	-0.00564913(1)	-0.006127581(2)	-0.00624132(5)	-0.0062572(5)
6×5	-0.0050962(5)	-0.00629315(2)	-0.00678613(2)	-0.00690240(9)	-0.0069568(9)
6×6	-0.0056530(6)	-0.00692336(2)	-0.00743660(2)	-0.0075559(4)	-0.0076436(9)
7×1	-0.0023249(1)	-0.003205140(1)	-0.003771247(3)	-0.00401211(9)	-0.0039591(1)
7×2	-0.0035857(4)	-0.004702613(2)	-0.005256009(3)	-0.00540519(9)	-0.0053248(2)
7×3	-0.0043928(4)	-0.005576718(2)	-0.006094720(8)	-0.0062148(1)	-0.0061856(3)
7×4	-0.0050434(4)	-0.00627435(2)	-0.00678550(2)	-0.0069021(1)	-0.0069285(3)
7×5	-0.0056347(5)	-0.00691997(2)	-0.00744009(2)	-0.0075593(3)	-0.0076344(4)
7×6	-0.0062004(5)	-0.00754866(3)	-0.00808495(4)	-0.0082071(5)	-0.0083245(7)
7×7	-0.0067540(6)	-0.00817135(3)	-0.00872727(5)	-0.0088536(5)	-0.009007(1)
8×1	-0.0026381(1)	-0.003633404(3)	-0.004269491(3)	-0.00453583(9)	-0.0044702(1)
8×2	-0.0040302(4)	-0.005275469(5)	-0.005884963(7)	-0.00604296(2)	-0.0059491(2)
8×3	-0.0048902(4)	-0.006190440(5)	-0.00675029(2)	-0.00687404(6)	-0.0068426(2)
8×4	-0.0055652(5)	-0.006898582(6)	-0.00744318(2)	-0.0075626(1)	-0.0075982(2)
8×5	-0.0061694(6)	-0.007545429(7)	-0.00809377(5)	-0.0082150(2)	-0.0083097(6)
8×6	-0.0067427(6)	-0.008172214(9)	-0.00873296(6)	-0.0088588(5)	-0.009002(1)
8×7	-0.0073012(6)	-0.00879187(2)	-0.00936927(7)	-0.0094986(9)	-0.009684(2)
8×8	-0.0078520(8)	-0.00940894(1)	-0.01000531(9)	-0.010138(1)	-0.010363(2)

TABLE VIII. Total values of V_W, V^a_{SW} and V^b_{SW}, Eq.(24), for various masses.

m	V_W	V^a_{SW}	V^b_{SW}
-1.014925	0.0034903(4)	-0.00196733(7)	0.00231391(3)
-0.9512196	0.0033381(1)	-0.0018394(1)	0.00228645(1)
-0.8749999	0.0031618(2)	-0.00168960(1)	0.00225268(4)
-0.8253968	0.0030498(1)	-0.00159403(3)	0.00223031(4)
-0.7948719	0.0029818(1)	-0.0015358(2)	0.00221643(3)
-0.5181059	0.0023900(8)	-0.0010308(6)	0.0020891(1)
-0.423462	0.0021944(9)	-0.0008671(8)	0.00204579(7)
-0.4157708	0.002179(1)	-0.000854(1)	0.00204228(6)
-0.4028777	0.0021523(4)	-0.000831(1)	0.00203642(2)
-0.3140433	0.001974(3)	-0.000696(2)	0.0019965(4)
-0.3099631	0.001966(2)	-0.000685(7)	0.0019947(5)
-0.301775	0.001951(6)	-0.000663(4)	0.0019914(6)
-0.2962964	0.001939(2)	-0.000653(2)	0.0019893(6)
-0.2852897	0.001915(6)	-0.000634(2)	0.0019844(5)
-0.2825278	0.001909(7)	-0.000629(1)	0.0019828(6)
-0.2769916	0.001898(3)	-0.000620(1)	0.0019798(5)
-0.2686568	0.001883(6)	-0.000607(4)	0.0019760(2)
-0.1482168	0.001634(8)	-0.000401(4)	0.0019227(2)
0.0	0.00134096(5)	-0.0001641(1)	0.00185871(2)
0.005	0.0013266(1)	-0.0001561(2)	0.00185656(1)
0.01	0.0013112(8)	-0.0001481(3)	0.001854398(7)
0.014	0.0012982(9)	-0.0001420(1)	0.001852668(1)
0.016	0.001291(1)	-0.0001389(1)	0.001851801(2)
0.018	0.001284(2)	-0.0001360(2)	0.001850934(5)
0.0236	0.0012623(4)	-0.0001279(2)	0.00184850(1)
0.027	0.0012499(7)	-0.0001235(2)	0.00184701(1)
0.035	0.001226(1)	-0.0001127(4)	0.00184349(4)
0.0366	0.001218(4)	-0.0001105(5)	0.00184278(4)
0.038	0.001212(4)	-0.0001087(6)	0.00184216(4)
0.0427	0.001194(4)	-0.0001032(2)	0.00184007(4)
0.046	0.001183(5)	-0.000100(1)	0.00183859(4)
0.0535	0.001155(3)	-0.000092(2)	0.00183522(4)
0.055	0.001150(3)	-0.000091(2)	0.00183454(4)
0.072	0.001100(3)	-0.000073(3)	0.00182676(4)
0.0927	0.001052(5)	-0.0000615(9)	0.00181709(5)

TABLE IX. $V_{Overlap}$, Eq.(26), as a function of M_O.

M_O	$V_{Overlap}$
0.01	-0.0004974(4)
0.05	-0.000477(4)
0.1	-0.000483(2)
0.2	-0.000517(3)
0.3	-0.000526(2)
0.4	-0.000533(1)
0.5	-0.000546(1)
0.6	-0.0005543(4)
0.7	-0.0005617(1)
0.8	-0.00056556(7)
0.9	-0.00056756(2)
1.	-0.00056833(7)
1.1	-0.00056827(3)
1.2	-0.0005681(4)
1.3	-0.0005690(1)
1.4	-0.000571(1)
1.5	-0.000573(3)
1.6	-0.000591(3)
1.7	-0.000612(5)
1.8	-0.000651(3)
1.9	-0.000629(3)
1.95	-0.000649(3)
1.99	-0.000616(1)

TABLE X. $F_{Clover}(R)$, Eq.(30), for specific m values ($c_{SW} = 1.3$).

R	$m = -0.5181059$	$m = 0.0$	$m = 0.038$
1	0.00232276(8)	0.00180277(2)	0.0017638(1)
2	0.0035644(3)	0.0030071(2)	0.0029552(2)
3	0.0039016(5)	0.0033731(2)	0.0033175(2)
4	0.0040550(7)	0.0035566(3)	0.0034997(7)
5	0.0041511(9)	0.0036759(4)	0.003618(1)
6	0.0042185(9)	0.0037608(5)	0.003702(1)
7	0.004268(1)	0.0038242(9)	0.003765(3)
8	0.00431(1)	0.003873(6)	0.003814(4)
9	0.00434(3)	0.00393(2)	0.00387(2)
10	0.00436(3)	0.00395(2)	0.00389(2)
11	0.00437(5)	0.00397(4)	0.00391(3)
12	0.00439(11)	0.00398(6)	0.00393(5)
13	0.0044(25)	0.00399(13)	0.00394(11)

TABLE XI. $F_{Overlap}(R)$, Eq.(30), for specific M_O values.

R	$M_O = 1.0$	$M_O = 1.3$	$M_O = 1.5$
1	-0.00049870(6)	-0.00052178(1)	-0.00052476(4)
2	-0.00063045(9)	-0.00064031(9)	-0.00063809(4)
3	-0.0006583(2)	-0.0006635(2)	-0.00066247(7)
4	-0.0006619(2)	-0.0006659(5)	-0.0006674(8)
5	-0.0006598(5)	-0.000663(1)	-0.0006664(8)
6	-0.0006563(6)	-0.000659(2)	-0.0006644(9)
7	-0.000650(3)	-0.000656(2)	-0.0006619(9)
8	-0.000648(3)	-0.000653(2)	-0.000659(1)
9	-0.000646(3)	-0.000651(3)	-0.000657(2)
10	-0.000644(3)	-0.000648(3)	-0.000655(5)
11	-0.000640(3)	-0.000646(4)	-0.000656(6)
12	-0.000639(6)	-0.000642(6)	-0.000653(7)
13	-0.000635(9)	-0.000641(8)	-0.000649(9)