Higher Loop Results for the Plaquette, Using the Clover and Overlap Actions

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Abstract

We calculate the perturbative value of the free energy in QCD on the lattice. This quantity is directly related to the average plaquette.

Our calculation is done to 3 loops using the clover action for fermions; the results are presented for arbitrary values of the clover coefficient, and for a wide range of fermion masses.

In addition, we calculate the 2 loop result for the same quantity, using the overlap action.

- We compute the perturbative expansion of the average plaquette, in $SU(N)$ gauge theory with N_f fermion flavours.

- We present separate calculations using the clover action (3-loops), and the overlap action (2-loops). The simpler case of Wilson fermions was performed in [1].

- The average plaquette can be related to the perturbative free energy of lattice QCD, as well as to the expectation value of the action.

- The results can be used:

  - In improved scaling schemes, using an appropriately defined effective coupling.

  - In long standing efforts, starting with [2], to determine the value of the gluon condensate.

  - In studies of the interquark potential [3].

  - As a test of perturbation theory, at its limits of applicability.

Calculation with Clover Fermions

▷ The action is, in standard notation:

$$S = S_W + S_f,$$
$$S_W = \beta \sum_{\Box} E_W(\Box), \qquad (1)$$
$$S_f = \sum_x E_f(x)$$

with

$$E_W(\Box) = 1 - \frac{1}{N} \mathrm{Re\ Tr}\,(\Box)\,,$$
$$\begin{aligned} E_f(x) = \sum_f \Big[(m+4r)\, \overline{\psi}^f_x \psi^f_x - \\ \frac{1}{2} \sum_\mu \Big(\overline{\psi}^f_{x+\hat\mu} (r+\gamma_\mu)\, U^\dagger_\mu(x) \psi^f_x + \overline{\psi}^f_x (r-\gamma_\mu)\, U_\mu(x) \psi^f_{x+\hat\mu} \Big) \Big] \\ + \frac{i}{4}\, c_{\mathrm{SW}} \sum_f \sum_{\mu,\nu} \overline{\psi}^f_x \sigma_{\mu\nu} \hat{F}^{\mu\nu}_x \psi^f_x . \end{aligned} \qquad (2)$$

where : $\hat{F}^{\mu\nu} \equiv \frac{1}{8} (Q_{\mu\nu} - Q_{\nu\mu}), \quad Q_{\mu\nu} = U_{\mu,\nu} + U_{\nu,-\mu} + U_{-\mu,-\nu} + U_{-\nu,\mu}$

Here $U_{\mu,\,\nu}(x)$ is the usual product of link variables $U_\mu(x)$ along a plaquette in the μ-ν directions, originating at x; f is a flavor index; m is the bare fermionic mass; $\sigma_{\mu\nu} = (i/2)[\gamma_\mu,\, \gamma_\nu]$; powers of a may be directly reinserted by dimensional counting. The clover coefficient c_{SW} is a free parameter in the present work; it is normally tuned in a way as to minimize $\mathcal{O}(a)$ effects.

▷ $\langle E_f \rangle$: straightforwardly computed to all orders, rescaling the action S_f by a factor ϵ under the fermionic path integral Z^f

$$Z^f(\epsilon) \equiv \int \mathcal{D}\overline{\psi}(x)\mathcal{D}\psi(x) \exp\left(-\epsilon S_f\right) = \epsilon^{4VNN_f} Z^f(\epsilon = 1) \tag{3}$$

and using

$$\langle E_f \rangle = -\frac{\partial}{\partial \epsilon}\left(\frac{\ln Z^f(\epsilon)}{V}\right)_{\epsilon=1} = -4NN_f. \tag{4}$$

▷ $\langle E_W \rangle$ is calculated in perturbation theory:

$$\langle E_W \rangle = 1 - \frac{1}{N}\langle \mathrm{Tr}\,(\Box) \rangle = c_1\, g^2 + c_2\, g^4 + c_3\, g^6 + \cdots \tag{5}$$

▷ The n-loop coefficient can be written as $c_n = c_n^g + c_n^f$ where:

- c_n^g : pure Yang-Mills contribution, known to 3-loops [4].
- c_n^f : fermionic contribution, known to 3-loops in the absence of the clover term [1].

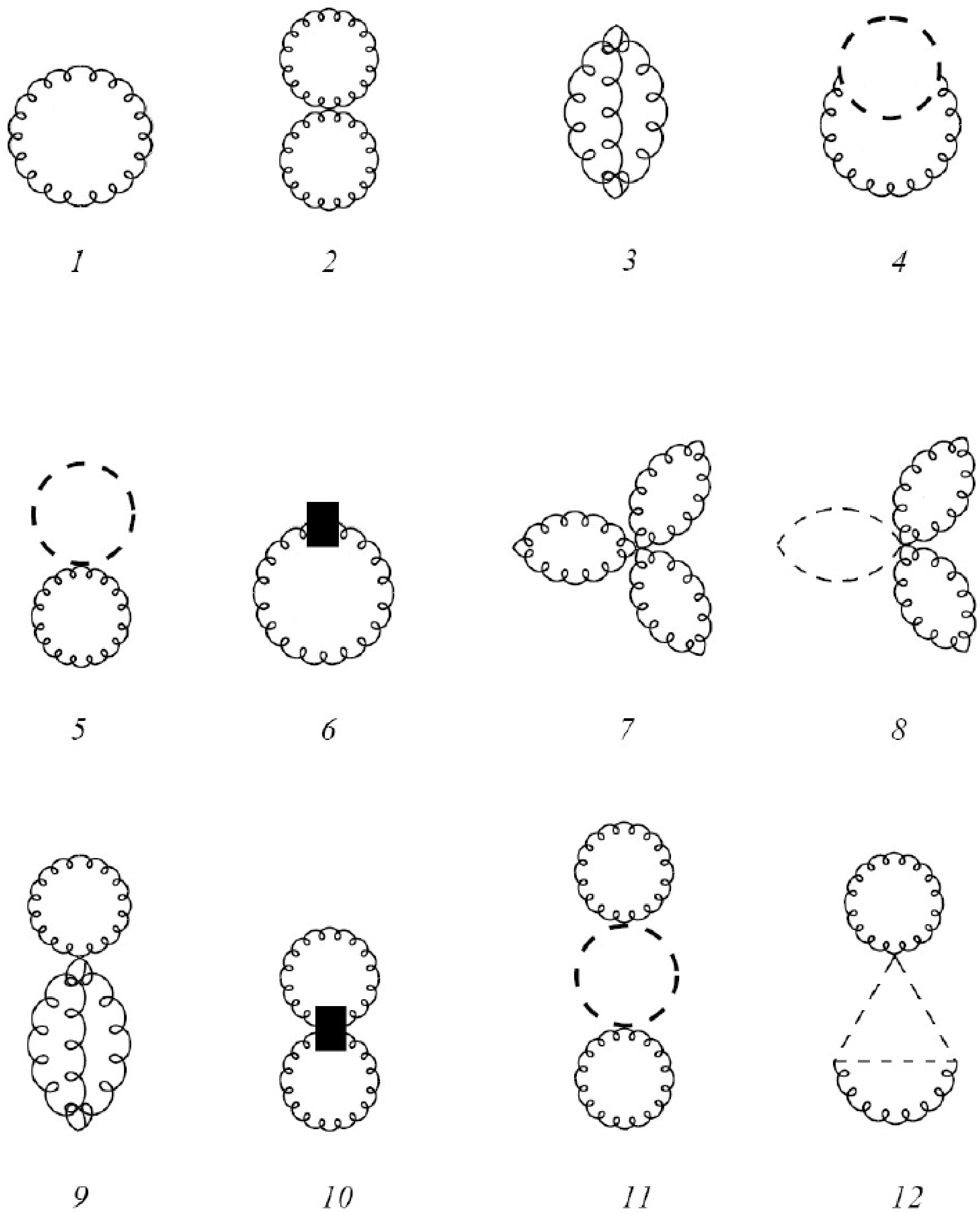
To calculate c_n we will first compute the free energy $-(\ln Z)/V$ up to 3 loops, Z being the full partition function

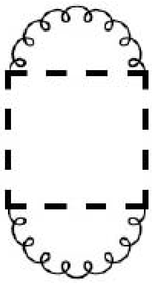
$$Z \equiv \int \mathcal{D}U_\mu(x)\mathcal{D}\overline{\psi}(x)\mathcal{D}\psi(x) \exp(-S). \tag{6}$$

The average of E_W is then extracted as follows

$$\langle E_W \rangle = -\frac{1}{6}\frac{\partial}{\partial \beta}\left(\frac{\ln Z}{V}\right). \tag{7}$$

Feynman diagrams for the free energy up to 3 loops





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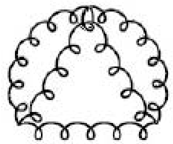

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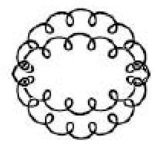

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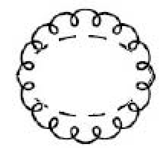

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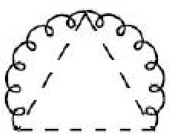

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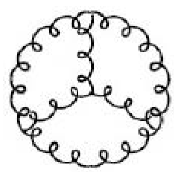

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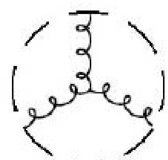

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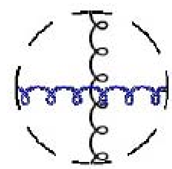

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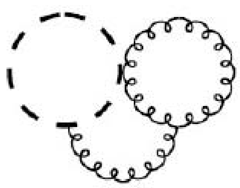

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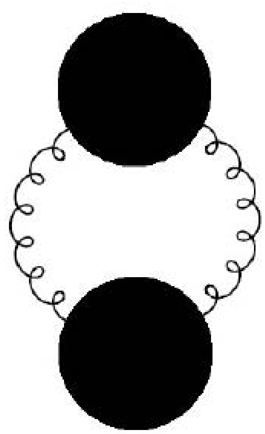

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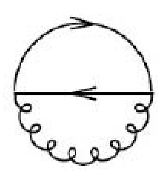

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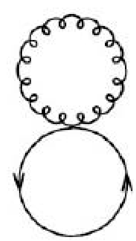

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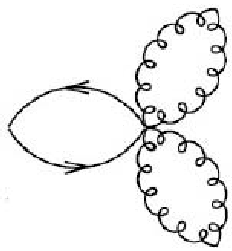

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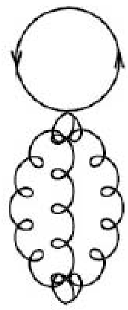

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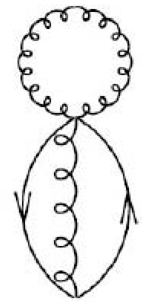

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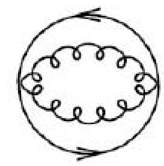

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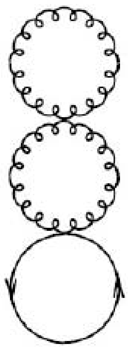

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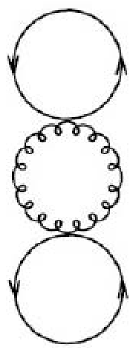

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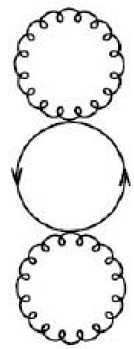

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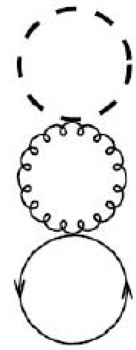

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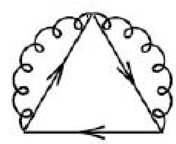

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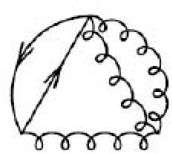

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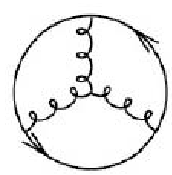

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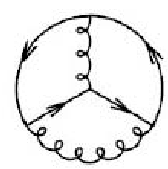

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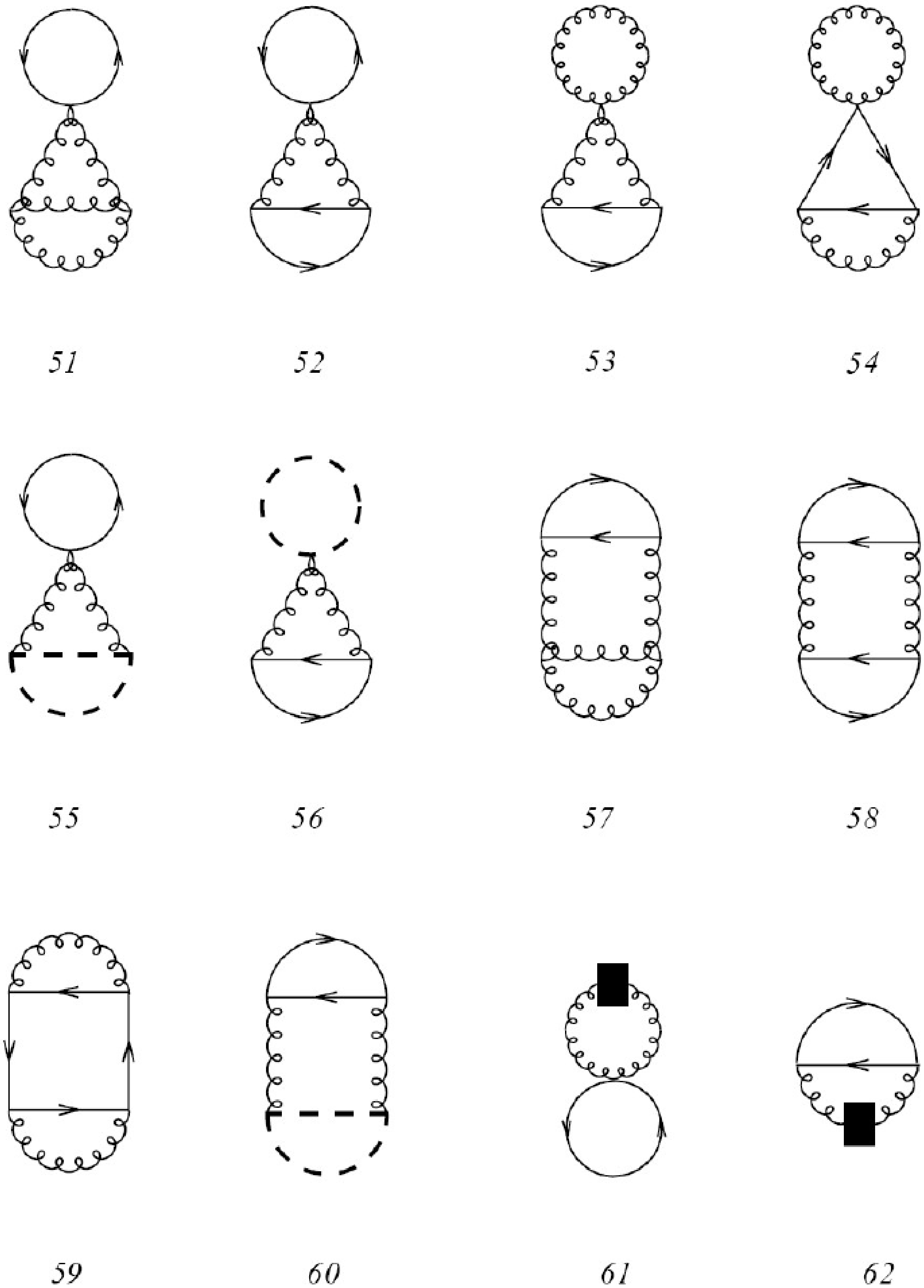
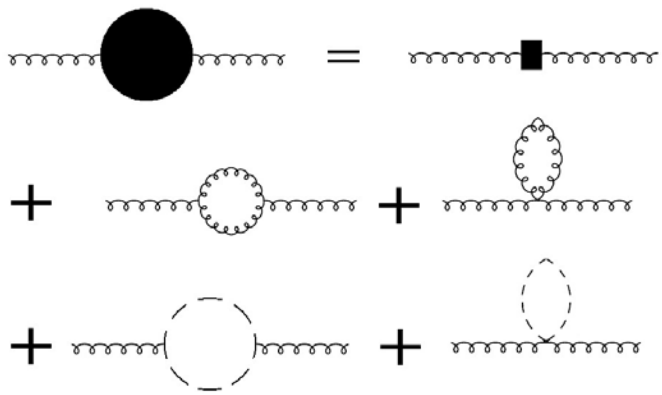


- ▷ Lines: Solid (fermions), curly (gluons), dashed (ghosts).
- ▷ Filled square: Contribution from the "measure" part of the action.

▷ Filled circle:



▷ The involved algebra of the lattice perturbation theory was carried out using our computer package in Mathematica.

▷ The value for each diagram is computed numerically for a sequence of finite lattice sizes. Diagrams must be grouped in several infrared-finite sets, before extrapolating their values to infinite lattice size.

▷ Extrapolation leads to a (small) systematic error, which is estimated quite accurately.

Results

▷ Pure gluonic contributions already known:

$$
\begin{aligned}
c_1^g &= \frac{N^2-1}{8\ N}, \\
c_2^g &= \left(N^2-1\right)\left(0.0051069297 - \frac{1}{128\ N^2}\right), \\
c_3^g &= \left(N^2-1\right)\Big(\frac{0.0023152583(50)}{N^3} - \frac{0.002265487(17)}{N} + \\
&\qquad 0.000794223(19)\ N\Big).
\end{aligned}
\tag{8}
$$

▷ Fermionic contributions take the form:

$$
\begin{aligned}
c_1^f &= 0 \, , \\
c_2^f &= \left(N^2-1\right) h_2 \, \frac{N_f}{N} , \\
c_3^f &= \left(N^2-1\right) \left(h_{30} \, N_f + h_{31} \, \frac{N_f}{N^2} + h_{32} \, \frac{N_f^2}{N} \right) .
\end{aligned}
\tag{9}
$$

$h_2, h_{30}, h_{31}, h_{32}$ depend polynomially on the clover parameter $c_{\rm SW}$:

$$
\begin{aligned}
h_2 &= h_2^{(0)} + h_2^{(1)} \, c_{\rm SW} + h_2^{(2)} \, c_{\rm SW}^2 \qquad (10) \\
h_{3i} &= h_{3i}^{(0)} + h_{3i}^{(1)} \, c_{\rm SW} + h_{3i}^{(2)} \, c_{\rm SW}^2 + h_{3i}^{(3)} \, c_{\rm SW}^3 + h_{3i}^{(4)} \, c_{\rm SW}^4
\end{aligned}
$$

▷ Our results for $h_2^{(j)}$, $h_{3i}^{(j)}$ are shown in the graphs below, for typical values of the bare mass m. Systematic errors are too small to be visible.

▷ As a typical example, setting $N = 3$, $N_f = 2$, and $m = -0.518106$ (corresponding to $\kappa = (8+2m)^{-1} = 0.1436$), we obtain:

$$
\begin{aligned}
c_{\rm SW} = 0: \quad \langle E_W \rangle &= (1/3) \, g^2 \; + 0.026185200(3) g^4 \; + 0.0119649(3) g^6 , \\
c_{\rm SW} = 2: \quad \langle E_W \rangle &= (1/3) \, g^2 \; + 0.013663456(3) g^4 \; + 0.0110200(13) g^6 .
\end{aligned}
\tag{11}
$$