Calculating the closed string spectrum of SU(N) gauge theories.

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1 Introduction.

- The main analytic method for calculating physics in complex field theories such as QCD is Pertubation Theory.
- Problem: Pertubation theory works well for problems where the first approximation is more-or-less a sensible description of what is going on.
  - In QED: First approximation is a free field theory of photons and electrically charched particles (electrons for instance), which has to do with the real world (electric currents).
  - In QCD: First approximation is a free field theory of quarks and gluons. But QCD is the theory of strong interactions describing a world of colourless particles like mesons, protons and other color singlet objects. So the first approximation is nothing at all like the real world of colourless hadrons. This is a non-perturbative problem!
- Lattice field theory provides a framework within which a numerical solution of the problem can be attempted.

2 Lattice Field Theory

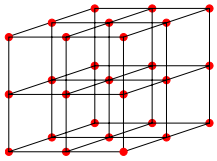
- We want to calculate the spectrum of a Hamiltonian. As we will see later, we can calculate this spectrum by calculating correlation functions of functionals of the fields. A correlator can be expressed as:

$$\langle \Phi^\dagger(t)\Phi(0)\rangle = \frac{1}{Z}\int \prod_{x,\mu} dA_\mu(x)\Phi^\dagger(t)\Phi(0)e^{iS} \tag{1}$$

  where: $S = \int dt \int d^3x\mathcal{L}$.
- We do not know how to calculate such integrals analytically for non perturbative quantities such as the mass spectrum, so we can attempt to calculate the integrals numerically.
- The presence of an oscillating factor destabilises the numerical approximation.
  - We calculate the correlation function in the Euclidean space-time $(t \to -it)$
  - $e^{iS} = e^{i\int dt \int d^3x\mathcal{L}} \to e^{-\int dt \int d^3x\mathcal{L}_\mathcal{E}} = e^{-S_E}$
- In an $SU(N)$ gauge theory the Euclidean correlator can be calculated from the corresponding Euclidean Feynman Path Integral:

$$\langle \Phi^\dagger(t)\Phi(0)\rangle_E = \frac{1}{Z}\int \prod_{x,\mu} dA_\mu(x)\Phi^\dagger(t)\Phi(0)e^{-\frac{1}{g^2}\int Tr\{F_{\mu\nu}^2\}d^4x} \tag{2}$$

- The above Path Integral involves an infinite number of degrees of freedom. We want to calculate the integral numerically so we must make the problem finite introducing ultraviolet and infrared cut-offs.
- We replace the continuous space-time by a (hyper)cubic lattice of points, of lattice spacing a and volume V (if $V = L^4$, $a \ll 1/\Lambda$ and $La \gg 1/\Lambda$ where Λ is the physical length scale of the problem):



- On the lattice, the gauge fields are replaced by some degrees of freedom caled links ($U_\mu(n)$ where n labels the lattice site and μ the direction of the link)
  - The gauge fields $A_\mu(x)$ belong to the $SU(N)$ Lie algebra.
  - The variables $U_\mu(x)$ belong to the $SU(N)$ group: $U_\mu(n) = e^{iagA_\mu^\alpha(n)\lambda^\alpha}$, where λ^α is a set of Hermitian matrices which generate the group and $A_\mu = A_\mu^\alpha\lambda^\alpha$.
- Now:

$$\langle \Psi_L(U)\rangle = \frac{1}{Z}\int \prod_{n,\mu} dU_\mu(n)\Psi_L(U)e^{-S_L[U]}\overset{a\to 0}{\longrightarrow}\langle\Psi(A)\rangle \tag{3}$$

  Where:

$$dU_\mu(n)\overset{a\to 0}{\longrightarrow}dA_\mu(x=an)$$

$$S_L[U]\overset{a\to 0}{\longrightarrow}S[A]$$

$$\Psi_L(U)\overset{a\to 0}{\longrightarrow}\Psi(A)$$

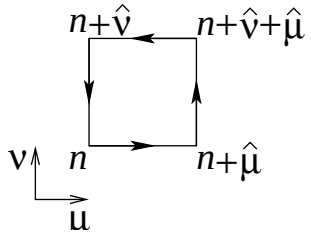
- Under a gauge transformation $G(n) \in SU(N)$:

$$U_\mu(n) \to G(n)U_\mu(n)G^\dagger(n+\hat{\mu}) \tag{4}$$

- A path ordered product of group elements along any closed path (that starts and ends at the site n) is gauge invariant. The simplest non-trivial

closed path is around the elementary square of the lattice. This is called a plaquette p.

$$\begin{aligned} TrU_p &= Tr[U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)] \\ &\xrightarrow{G.T} Tr[U_p] \end{aligned} \tag{5}$$



- The action takes the following form:

$$S = \beta \sum_p \{1 - \frac{1}{N_c} ReTrU_p\} \tag{6}$$

where: $\beta = \frac{2N_c}{g^2}$

3 Monte-Carlo simulations

- We want to calculate the expectation value of colour singlet operators.
- High multidimensionality of these integrals makes traditional mesh teqniques impractical.

  For example, consider a 10^4 site lattice. This system has 40000 link variables. Let us take the simplest possible gauge group Z_2. Choosing this gauge group the partition function becomes an ordinary sum which has an enormous number of terms ($2^{40000} = 1.58 \times 10^{12041}$)! The appearance of such large numbers immediately suggests a statistical treatment.

  $\rightarrow$ Monte Carlo methods.

- We need to generate n_c different field configurations with probability distribution:

$$\prod_l dU_l e^{-\beta \sum_p \{1 - \frac{1}{N_c} ReTrU_p\}} \tag{7}$$

- Then the expectation value of $\Psi_L(U)$ will be just the average over these fields.

$$\langle \Psi_L(U) \rangle = \frac{1}{n_c} \sum_{I=1}^{n_c} \Psi_L(U^I) \pm \mathcal{O}(\frac{1}{\sqrt{n_c}}) \tag{8}$$

4 Large-N_c Field Theories

- It was first pointed out by 't Hooft that many features of QCD can be understood by studying a gauge theory based on the gauge group $SU(N)$ in the limit $N \to \infty$.
- Two questions:

  ◇ Does $N \to \infty$ limit make the analysis more complicated? (more dynamical degrees of freedom).

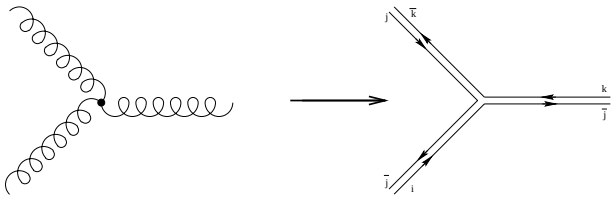
  ◇ How much is $SU(N)$ related to $SU(3)$?
- $SU(N)$ gauge theory simplifies in the $N \to \infty$ where the expansion parameter is $1/N$.
- Results for $SU(3)$ can be obtained from the $N \to \infty$ limit by expanding in $1/N = 1/3$.

  ◇ Is $1/N$ a small enough expansion parameter for QCD?

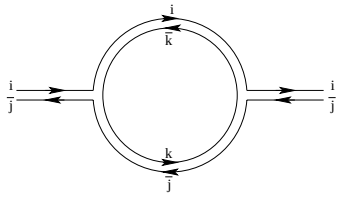
  ◇ In QED the coupling constant is equal to $e = \sqrt{4\pi\alpha} = 0.30$, which is not so different from $1/3 = 0.333....$
- Examples:

  ◇ 't Hooft's coupling: $\lambda = g^2 N_c$

  ◇ 't Hooft's double line diagrammatic representation:

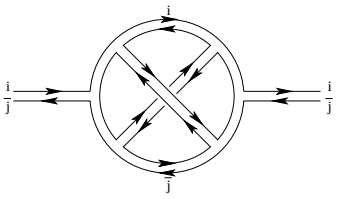


  ◇ Planar diagram:



$$\sim g^2 \times N_c(1 \text{ closed loop}) = \frac{\lambda}{N_c} \times N_c = \lambda$$

  ◇ Non-Planar diagram:



$$\sim g^6 \times N_c(1 \text{ closed loop}) = \frac{\lambda^3}{N_c^3} \times N_c = \frac{\lambda^3}{N_c^2} \overset{N_c \to \infty}{\longrightarrow} 0$$

5 SU(N_c) and Strings

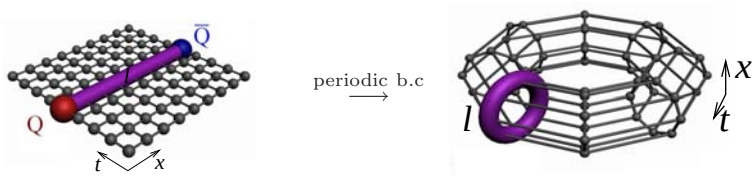
- What string theory describes $SU(N_c = \infty)$?

  It is an old idea that large-N QCD might be exactly reformulated as a string theory. This connection has never been made precise, but the topological structure of the pertubation theory makes it plausible that one could first recast it as a string theory, and then apply string methods to determine the spectrum and amplitudes.

- Effective string theory models describe the confining flux tube.

  A Flux tube is the storage medium for the linearly rising interquark potential. The basic assumption of this model is that at low spatial resolution, a (lattice spacing), the quantum field theory of the gluon fields, can be modelled by the quantum mechanics of a string like object (this model involves mechanical degrees of freedom only), the flux tube.

  A quark-antiquark pair will experience an attractive force which remains non-vanishing even for asymptotically large separations. This linearly increasing long-distance potential energy form the basis of essentially all models of quark confinement.



$$\Phi(l,t) = \phi^{\dagger}(0,t)U(0,l;t)\phi(l,t) \qquad \Phi(l,t) = TrU(l;t)$$

- Nambu-Goto effective string theory for a closed string (winding flux tube) gives the following relation:

$$E^2_{n,N+\tilde{N}} = \sigma^2 l^2 w^2 - \frac{\pi}{3}\sigma(D-2) + 4\pi\sigma(N+\tilde{N}) + \vec{p}^{\,2}_{n||} + \vec{p}^{\,2}_{n\perp} \qquad (9)$$

  where:

  $N + \tilde{N}$ describes the sum of right and left movers (String theory),

  w is the winding number,

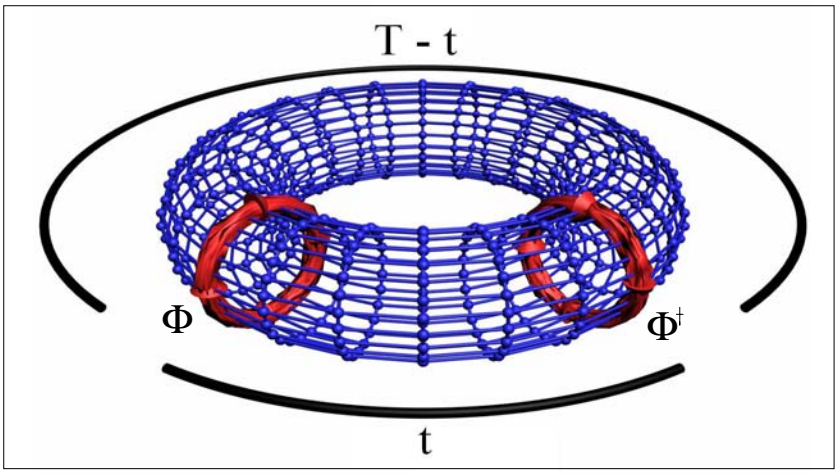
  $\vec{p}^{\,2}_{n||} = (2\pi n/l)^2$ (n=0,1,2, ...) is the momentum square along the compactified dimension of length l,

  $\sigma \simeq (440 \pm 30 MeV)^2$ is the string tension, the coefficient of the linear part of the confining potential which dominates at large separations between charges of colour (e.g. quarks) ($V(l) \overset{t\to\infty}{=} \sigma l$). If one uses a potential that is the sum of Columb and linear terms in the calculation of the charmonium spectrum one finds that to reproduce the experimental spectrum one needs the above value of σ. The error is large due to the inherent ambiguity in trying to introduce physical MeV units into the calculation.

6 Energy calculation

- Masses of certain states can be calculated, using the correlation functions of specific operators:

$$\begin{aligned} \langle \Phi^\dagger(t)\Phi(0)\rangle &= |\langle\Omega|\Phi^\dagger|0\rangle|^2 e^{-tm_0} + \sum_{m\geq 1} |\langle\Omega|\Phi^\dagger|m\rangle|^2 e^{-tm_m} \\ &\stackrel{t\to\infty}{\longrightarrow} |\langle\Omega|\Phi^\dagger|0\rangle|^2 e^{-tm_0} \end{aligned} \tag{10}$$



- Let us define the effective mass:

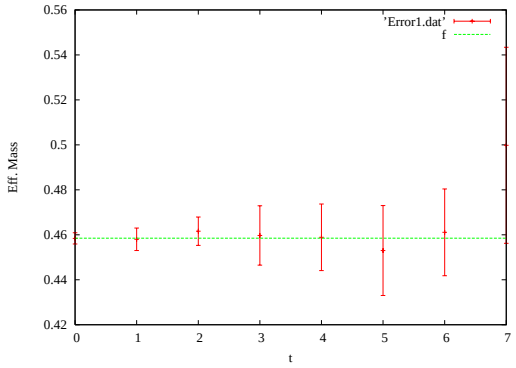
$$am_{eff}(t) = -\ln \frac{\langle \Phi^\dagger(t)\Phi(0)\rangle}{\langle \Phi^\dagger(t-a)\Phi(0)\rangle} \tag{11}$$

- The mass will be equal to:

$$am_0 \simeq am_{eff}(t_0) \tag{12}$$

  where t_0 is the lowest value of t for which $m_{eff}(t_0) = m_{eff}(t > t_0)$

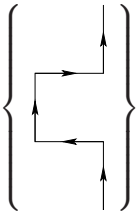
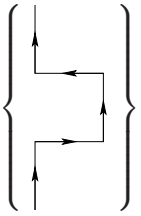
- For examle:



- The basis of the operators is not complete.
- Using this procedure it is impossible to extract the masses of the excited states.
- We need to construct 'good' operators.
- The best operator would be the one that maximises:

$$C(t) = \frac{\langle \Phi^\dagger(t)\Phi(0)\rangle}{\langle \Phi^\dagger(0)\Phi(0)\rangle} \tag{13}$$

- So, we must use variational calculation!
  - We expect to calculate masses for some excited states, so we need to use operators which merely project onto the excited states.
  - We construct a basis of operators, $\phi_i : i = 1, ..., N_O$, with transverse deformations described by the quantum numbers of parity, winding number, longitudinal momentum and transverse momentum. For example:
    If

$\phi_L = \mathrm{Tr}$  and $\phi_\mathrm{R} = \mathrm{Tr}$  (14)

    then:

$$\Phi_\pm^{p_x,p_y} = \frac{1}{L_x L_y}\sum_{x,y}\{\phi_L \pm \phi_R\}\, e^{ip_x x + ip_y y} \tag{15}$$

– We calculate the correlation function (Matrix):

$$C_{ij}(t) = \langle \phi_i^\dagger(t)\phi_j(0)\rangle \tag{16}$$

– We diagonalize the matrix: $C^{-1}(0)C(a)$.

– We extract the correlator for each state.

– By fitting the results, we extract the mass (energy) for each state.