Group Theory

Postgraduate Lecture Course

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Fisher Room, Denys Wilkinson Building

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Outline

1) Groups and representations
2) Finite groups
3) Lie groups
   a) Lie groups
   b) Lie algebras
4) Examples
   a) Lorentz and Poincaré group
   b) $SU(n)$ and tensor methods
   c) $SO(n)$, spinor representations
5) Classification of simple Lie algebras
6) Representations and Dynkin formalism

Course information

Modern theories of particle physics are based on symmetry principles and use group theoretical tools extensively. Besides the standard Poincaré/Lorentz invariance of all such theories, one encounters internal (continuous) groups such as $SU(3)$ in QCD, $SU(5)$ and $SO(10)$ in grand unified theories (GUTs), and E_6 and E_8 in string theory. Discrete groups also play an important role in particle physics model building, for example in the context of models for fermion masses. The main purpose of this course is to develop the understanding of groups and their representations, including finite groups and Lie groups. Emphasis is placed on a mathematically satisfactory exposition as well as on applications to physics and practical methods needed for "routine" calculations.

Literature

- W. Fulton and J. Harris, "Representation Theory", Springer, Graduate Texts in Mathematics.
- T. Bröcker and T. tom Dieck, "Representations of Compact Lie Groups", Springer, Graduate Texts in Mathematics.
- H. Boerner, "Representations of groups: with special consideration for the needs of modern physics", North-Holland Pub. Co., 1963.
- R. Slansky, "Group Theory for Unified Model Building", *Phys. Rep.* **79** (1981) 1.
- B. G. Wybourne, "Classical Groups for Physicists", Wiley (1974).
- M. Gourdin, "Unitary Symmetries", North Holland (1967).
- R. N. Cahn, "Semi-Simple Lie Algebras and Their Representations", Benjamin/Cummings (1984).
- H. Georgi, "Lie Algebras in Particle Physics", Benjamin/Cummings (1982).

Prerequisites

The course assumes knowledge of

- linear algebra
- the groups $SO(3)$ and $SU(2)$, as, for example, encountered in the context of quantum mechanics

It will help to have come across the following subjects

- some basic differential geometry
- the $SU(3)$ quark model
- gauge-field theories

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Problem Sheet 1

Date: Nov 19, 2013, Deadline: Dec 3, 2013

1) (Generalisation of Schur's Lemma) Write the reducible representation R of G as $R = n_1 R_1 \oplus \cdots \oplus n_r R_r$ where R_i, $i = 1, \cdots, r$ are irreducible representations of dimensions d_i and the integers n_i indicate how often R_i appears in R. Convince yourself that the representation matrices $R(g)$ can then be written as $R(g) = \mathbf{1}_{n_1} \otimes R_1(g) \oplus \cdots \oplus \mathbf{1}_{n_r} \otimes R_r(g)$. (Here, the tensor product $A \otimes B$ of two matrices A and B denotes the matrix obtained when every entry of A is replaced by this entry times the matrix B). Then, show that a matrix P with $[P, R(g)] = 0$ for all $g \in G$ has the general form $P = P_1 \otimes \mathbf{1}_{d_1} \oplus \cdots \oplus P_r \otimes \mathbf{1}_{d_r}$ where P_i are $n_i \times n_i$ matrices.

2) (Permutation groups) Denote by S_n the group of permutations of n objects, that is $S_n = \{\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \,|\, \sigma \text{ bijective}\}$. It is often useful to denote a particular permutation σ by the symbol

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} .$$

   a) Verify that S_n forms a group for all n which is non-Abelian for $n > 2$.
   b) Focus on S_3. Determine its conjugacy classes and show that the complete set of its complex irreducible representations consists of one two-dimensional and two one-dimensional representations.
   c) Find the character table of S_3.
   d) Consider the regular representation of S_3 and write down the projectors which correspond to the various irreducible representations.

3) (Left-invariant one-forms and Maurer Cartan equation) For a matrix Lie group, consider the left-invariant vectors fields $L_i = {\xi_i}^j \frac{\partial}{\partial t^j}$ and the dual one-forms $\phi^i = {\phi_j}^i dt^j$ where ${\phi_i}^j {\xi_j}^k = \delta_i^k$.

   a) Using the results on left-invariant vector fields from the lecture, show that $g^{-1} dg = \phi^i T_i$, where T_i are the generators.
   b) Use the result from a) to show that $d\phi^i + \frac{1}{2} {f_{jk}}^i \phi^j \wedge \phi^k = 0$, where ${f_{jk}}^i$ are the structure constants.

4) (Lie-groups and their Lie-algebras)

   a) Derive the Lie-algebras of $SO(4)$ and $SU(2) \times SU(2)$ and show that they are isomorphic.

b) Do the same for $SO(6)$ and $SU(4)$ (Hint: It is helpful to contruct a basis for the $SU(4)$ Lie algebra starting with gamma matrices in six Euklidean dimensions (these are 8×8 matrices) and their antisymmetrized products.)

c) Show that the $2n \times 2n$ real matrices M satisfying $M^T \eta M = \eta$ where

$$\eta = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

form a group. This group is called the symplectic group $Sp(2n)$. Find the Lie-algebra $sp(2n)$ of $Sp(2n)$ and its Cartan subalgebra. Further, determine $\dim(sp(2n))$ and $\text{rank}(sp(2n))$.

5) (The Lorentz group)
A Dirac spinor ψ transforms in the representation $R_D = (1/2, 0) \oplus (0, 1/2)$ of the Lorentz group and can be written as

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

where χ_L and χ_R are left- and right-handed Weyl spinors. The representation matrices $R_D(M)$ acting on ψ are given by

$$R_D(M) = \begin{pmatrix} R_L(M) & 0 \\ 0 & R_R(M) \end{pmatrix} .$$

Define the gamma matrices γ_μ by

$$\gamma_0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} , \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} .$$

a) Using the explicit expressions for $R_L(M)$ and $R_R(M)$, show that an infinitesimal transformation of ψ takes the form $\delta\psi = i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi$ where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $\epsilon^{\mu\nu}$ are small parameters.

b) Show explicitly that the matrices $\sigma_{\mu\nu}$ form a representation of the Lorentz group Lie algebra.

c) Use the relation between the Lorentz group and $SL(2, C)$ to show that $R_D(M)^{-1}\gamma_\mu R_D(M) = R_V(M)_\mu{}^\nu \gamma_\nu$.

d) Proof that the Dirac equation for the spinor ψ with mass m is Lorentz-covariant by applying the result c).

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Problem Sheet 2

Date: 2/12/2013, Deadline: 13/1/2013

1) ($SU(5)$, tensor methods and Grand Unification)

   a) Find the Young-tableaux and associated tensors for the representations $\mathbf{1}$, $\mathbf{5}$, $\bar{\mathbf{5}}$, $\mathbf{10}$, $\mathbf{15}$ and $\mathbf{24}$ of $SU(5)$.

   b) Show that

$$\begin{aligned} \mathbf{5} \times \bar{\mathbf{5}} &= \mathbf{1} + \mathbf{24} \\ \mathbf{5} \times \mathbf{5} &= \mathbf{10} + \mathbf{15} \\ \bar{\mathbf{5}} \times \mathbf{10} &= \mathbf{5} + \mathbf{45} \\ \mathbf{10} \times \mathbf{10} &= \mathbf{5} + \mathbf{45} + \mathbf{50} \end{aligned}$$

   using Young-tableaux.

   c) Using the obvious embedding of $SU(3) \times SU(2)$ into $SU(5)$ (such that, $U_3 \in SU(3)$ and $U_2 \in SU(2)$ are embedded as

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & U_2 \end{pmatrix} \in SU(5))$$

   show that one family of standard model particles exactly fits into the representations $\bar{\mathbf{5}}$ and $\mathbf{10}$. Identify the generator in the Cartan subalgebra of $SU(5)$ (in the $\mathbf{5}$ representation) which corresponds to weak hypercharge $U_Y(1)$.

   d) Assume there is a Higgs boson transforming in the $\bar{\mathbf{5}}$ representation. Write down the allowed ($SU(5)$-invariant) Yukawa couplings. Work out the pattern of fermion masses that arises when the $SU_W(2)$ doublet within the $\bar{\mathbf{5}}$ Higgs acquires a VEV.

2) (Grand Unification Lie-groups and their subgroups)
Using (extended) Dynkin diagrams, convince yourself that

$$\begin{aligned} E_8 &\supset SO(16); SU(5) \times SU(5); SU(3) \times E_6; SU(2) \times E_7; SU(9); SU(4) \times SO(10) \\ E_6 &\supset SO(10) \times U(1); SU(2) \times SU(6); SU(3) \times SU(3) \times SU(3) \\ SO(10) &\supset SU(5) \times U(1); SU(2) \times SU(2) \times SU(4); SO(8) \times U(1) \end{aligned}$$

3) (Dynkin formalism and Grand Unification)
Consider the chain $SO(10) \supset SU(5) \times U(1) \supset SU_c(3) \times SU_W(2) \times U_Y(1) \times U(1)$.

a) Construct the weight system of the $SO(10)$ representation with highest weight (00001). Show that this is the **16** representation and, hence, the spinor of $SO(10)$.

b) Construct the weight systems of the representations $\bar{\mathbf{5}} \sim (0001)$ and $\mathbf{10} \sim (0100)$ of $SU(5)$.

c) Verify the result from 1c) that $\bar{\mathbf{5}}$ and **10** of $SU(5)$ contain one standard model family by using the weight systems of the various representations involved and the projection matrix

$$P(SU(5) \supset SU_W(2) \times SU_c(3)) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

Also, show that weak hypercharge can be represented by $Y = [-2, 1, -1, 2]/3$ in the dual basis.

d) Find the decomposition of **16** of $SO(10)$ into $SU(5)$ representations and identify the standard model states within **16**. To do this, use the projection matrix

$$P(SO(10) \supset SU(5)) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

together with the above results. Which additional state do you find and what is its representation under the standard model group and its physical interpretation?

4) (Value of Casimir operator in Dynkin formalism)
Consider the representations $\mathbf{n} \sim (1, 0, \cdots, 0)$, $\bar{\mathbf{n}} \sim (0, \cdots, 0, 1)$ and $\mathbf{n^2 - 1} \sim (1, 0, \cdots, 0, 1)$ of $SU(n)$.

a) Compute the value of the quadratic Casimir C for those representations.

b) Compute the index c of those representations and determine the one-loop β–function for an $SU(n)$ Yang-Mills theory with N_f Dirac fermions in $\mathbf{n}$. Discuss the qualitative behaviour of the gauge coupling as a function of the energy scale for $N_f = 6$.

(Hint: The explicit form of the Cartan martices, metric tensors and much more can be found in R. Slansky, *Phys. Rep.* **79** (1981) 1.)