Vectors and Matrices

aka Linear Algebra

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course material at:
http://www-thphys.physics.ox.ac.uk/people/AndreLukas/V&M/

1) Vector spaces and vectors

Def. of vector space:

- vectors in V, scalars in field $F = \mathbb{R}, \mathbb{C}, \ldots$
- two operations: vector addition and scalar multiplication . . .
- . . . subject to a number of rules

Def. of sub-vector space:

Non-empty sub-set $W \subset V$ "closed" under vector addition and scalar multiplication.

"Lines, planes etc. through $\mathbf{0} \in V$"

Key examples for vector spaces:

$\mathbb{R}^n$, $\mathbb{C}^n$, $n \times m$ matrices, function vector spaces

Key concepts in vector spaces:

- **Linear combination:** $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i$

$$\boxed{\alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ -2\alpha + 3\beta \end{pmatrix}}$$

- **Span:** $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) := \left\{ \sum_{i=1}^{k} \alpha_i \mathbf{v}_i \mid \alpha_i \in F \right\}$

$$\boxed{\mathrm{Span}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) = \left\{ \begin{pmatrix} \alpha + 2\beta \\ -2\alpha + 3\beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}}$$

The span is a (sub) vector space.

- **Linear independence:** $\sum_{i=1}^{k} \alpha_i \mathbf{v}_i = \mathbf{0} \implies \text{all } \alpha_i = 0$

$$\alpha \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ -2\alpha + 3\beta \end{pmatrix} \overset{!}{=} \mathbf{0} \implies \alpha = \beta = 0$$

$\implies$ linearly independent

$$\alpha \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} \alpha + 3\beta + \gamma \\ -\alpha - 4\beta - 2\gamma \\ 2\alpha - 4\gamma \end{pmatrix} \overset{!}{=} \mathbf{0}$$

solved for $\alpha = 2,\ \beta = -1,\ \gamma = 1 \implies$ linearly dependent

$$\alpha e^x + \beta e^{2x} \overset{!}{=} 0 \qquad \begin{array}{lcl} x = 0 & : & \alpha + \beta = 0 \\ x = 1 & : & e\alpha + e^2\beta = 0 \end{array} \implies \alpha = \beta = 0$$

$\implies$ e^x, e^{2x} linearly independent

- **Basis:** $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is basis if it is lin. indep. and spans V

Standard unit vectors in $\mathbb{R}^n$, $\mathbb{C}^n$

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}}\text{component}$$

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^{n} v_i \mathbf{e}_i$$

$$\sum_{i=1}^{n} \alpha_i \mathbf{e}_i \overset{!}{=} \mathbf{0} \implies \text{all } \alpha_i = 0$$

$$\implies \mathbf{e}_1, \ldots, \mathbf{e}_n \text{ basis of } \mathbb{R}^n, \mathbb{C}^n$$

Standard unit vectors in $\mathbb{R}^3$, $\mathbb{C}^3$

$$\mathbf{i} = \mathbf{e}_1,\ \mathbf{j} = \mathbf{e}_2,\ \mathbf{k} = \mathbf{e}_3$$

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ x \end{pmatrix} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\,\mathbf{k} \overset{!}{=} \mathbf{0} \implies \alpha = \beta = \gamma = 0$$

$$\implies \mathbf{i},\ \mathbf{j},\ \mathbf{k} \text{ basis of } \mathbb{R}^3, \mathbb{C}^3$$

- Every vector is a unique linear combination of a basis.

Vector as linear combination of basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \ \text{basis of } \mathbb{R}^2$$

$$\mathbf{v} := \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \begin{pmatrix} \alpha + 2\beta \\ -\alpha - 3\beta \end{pmatrix}$$

$$\Longrightarrow \begin{array}{r} \alpha + 2\beta = 2 \\ -\alpha - 3\beta = -1 \end{array} \quad \Longrightarrow \quad \alpha = 4, \ \beta = -1$$

- Dimension of V: number of vectors in a basis for V

$$\dim_{\mathbb{R}}(\mathbb{R}^n) = \dim_{\mathbb{C}}(\mathbb{C}^n) = n$$

$$\text{real } n \times m \text{ matrices : dimension} = nm$$

2) Vector spaces $\mathbb{R}^n$, geometrical applications

- Scalar (dot) product in $\mathbb{R}^n$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad \mathbf{a} \cdot \mathbf{b} := \sum_{i=1}^{n} a_i b_i \ = \delta_{ij} a_i b_j$$

Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$

Length of vector: $|\mathbf{a}| := \sqrt{\mathbf{a} \cdot \mathbf{a}} = \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2}$

Angle between two vectors: $\cos(\sphericalangle(\mathbf{a}, \mathbf{b})) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \quad \Longrightarrow$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\sphericalangle(\mathbf{a}, \mathbf{b}))$$

$$\mathbf{a},\ \mathbf{b} \neq \mathbf{0} \text{ perpendicular } \iff \mathbf{a} \cdot \mathbf{b} = 0 \iff \sphericalangle(\mathbf{a}, \mathbf{b}) = \frac{\pi}{2}$$

Angle between two vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad |\mathbf{a}| = |\mathbf{b}| = \sqrt{6}\,, \quad \mathbf{a}\cdot\mathbf{b} = 3$$

$$\cos(\sphericalangle(\mathbf{a},\mathbf{b})) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{1}{2} \quad \Longrightarrow \quad \sphericalangle(\mathbf{a},\mathbf{b}) = \frac{\pi}{3}$$

• Vector product in $\mathbb{R}^3$

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} := \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \quad \textbf{or} \quad (\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_jb_k$$

Levi-Civita tensor: $\epsilon_{ijk} = \begin{cases} +1 & \text{if} \quad (i,j,k) = (1,2,3),(2,3,1),(3,1,2) \quad \text{“cyclic permutations”} \\ -1 & \text{if} \quad (i,j,k) = (2,1,3),(3,2,1),(1,3,2) \quad \text{“anti-cyclic permutations”} \\ \;\;0 & \quad\;\; \text{otherwise} \end{cases}$

Calculating vector products

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \color{red}{4} \\ \color{blue}{1} \\ \color{green}{2} \end{pmatrix}$$

Proving vector identities with indices

$$(\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))_i = \epsilon_{ijk}a_j(\mathbf{b}\times\mathbf{c})_k = \epsilon_{ijk}\epsilon_{kmn}a_jb_mn_n = \epsilon_{kij}\epsilon_{kmn}a_jb_mc_n \overset{(2.16)}{=} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})a_jb_mc_n$$
$$= a_jc_jb_i - a_jb_jc_i = \mathbf{a}\cdot\mathbf{c}b_i - \mathbf{a}\cdot\mathbf{b}c_i = ((\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c})_i$$

$$(\nabla\times(\nabla f))_i = \epsilon_{ijk}\partial_j(\nabla f)_k = \epsilon_{ijk}\partial_j\partial_k f = 0$$

Geometrical interpretation:

$$|\,\mathbf{a}\times\mathbf{b}\,| = |\,\mathbf{a}\,|\cdot|\,\mathbf{b}\,|\sin\sphericalangle(\mathbf{a},\mathbf{b}) = \text{area of parallelogram defined by } \mathbf{a},\mathbf{b}$$

$$\mathbf{a},\mathbf{b} \perp \mathbf{a}\times\mathbf{b}$$

- Triple product in $\mathbb{R}^3$

$$\begin{aligned}\langle\mathbf{a},\mathbf{b},\mathbf{c}\rangle &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det(\mathbf{a},\mathbf{b},\mathbf{c}) = \epsilon_{ijk}a_ib_jc_k\\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1\\ &= \text{ volume of parallelepiped defined by } \mathbf{a},\mathbf{b},\mathbf{c}\end{aligned}$$

$\Rightarrow$ vanishes if two arguments are equal

$$\det(\mathbf{a},\mathbf{b},\mathbf{c}) \neq 0 \quad \Longleftrightarrow \quad \mathbf{a},\mathbf{b},\mathbf{c} \text{ basis of } \mathbb{R}^3$$

• Lines in $\mathbb{R}^3$

vector form | Cartesian form

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \mathbf{p} + t\mathbf{q} \quad \longleftrightarrow \quad t = \frac{x - p_x}{q_x} = \frac{y - p_y}{q_y} = \frac{z - p_z}{q_z}$$

<u>Line through two points</u> $\quad A = (1, 2, -3)^T,\ B = (2, -1, 1)^T$

$$\Longrightarrow \quad \mathbf{p} = A = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{q} = B - A = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \quad \longleftrightarrow \quad t = \frac{x-1}{1} = \frac{y-2}{-3} = \frac{z+3}{4}$$

Minimal distance of line from point $\mathbf{p}_0$: $\quad d_{\min} = \frac{|(\mathbf{p} - \mathbf{p}_0) \times \mathbf{q}|}{|\mathbf{q}|}$

Minimal distance of two lines $\mathbf{r}_i(t_i) = \mathbf{p}_i + t_i\mathbf{q}_i$:

$$d_{\min} = |(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{n}| \qquad \mathbf{n} = \frac{\mathbf{q}_1 \times \mathbf{q}_2}{|\ \mathbf{q}_1 \times \mathbf{q}_2\ |}$$

• Planes in $\mathbb{R}^3$

vector form $\qquad \mathbf{n} = \dfrac{\mathbf{q}\times\mathbf{s}}{|\,\mathbf{q}\times\mathbf{s}\,|} \qquad$ **Cartesian form**

$$\mathbf{r}(t_1,t_2) = \begin{pmatrix} x(t_1,t_2) \\ y(t_1,t_2) \\ z(t_1,t_2) \end{pmatrix} = \mathbf{p} + t_1\mathbf{q} + t_2\mathbf{s} \quad \xleftrightarrow{d\,=\,\mathbf{n}\cdot\mathbf{p}} \quad \begin{aligned} &\mathbf{n}\cdot\mathbf{r} = d \\ &n_x x + n_y y + n_z z = d \end{aligned}$$

Plane through three points $\qquad A=(2,1,0)^T,\ B=(-1,2,-1)^T,\ C=(3,-2,4)^T$

$$\mathbf{p} = A = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{q} = B - A = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{s} = C - A = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

$$\mathbf{q}\times\mathbf{s} = (1,11,8)^T$$

$$\mathbf{p}\cdot(\mathbf{q}\times\mathbf{s}) = 13$$

$$\mathbf{r}(t_1,t_2) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_1\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \longleftrightarrow x + 11y + 8z = 13$$

Intersection of line $\mathbf{r}_L(t) = \mathbf{a} + t\mathbf{b}$ and plane $\mathbf{r}_P(t_1,t_2) = \mathbf{p} + t_1\mathbf{q} + t_2\mathbf{s}$:

$$t_{\text{isec}} = \frac{\langle \mathbf{p}-\mathbf{a}, \mathbf{q}, \mathbf{s}\rangle}{\langle \mathbf{b}, \mathbf{q}, \mathbf{s}\rangle}$$ **with intersection point at $\mathbf{r}_L(t_{\text{isec}})$**

3) Linear maps and matrices

Def. of linear map $f : V \to W$:

$$\textit{(L1)}\ f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$$
$$\textit{(L2)}\ f(\alpha \mathbf{v}) = \alpha f(\mathbf{v})$$

Kernel and image of a linear map (sub vector spaces):

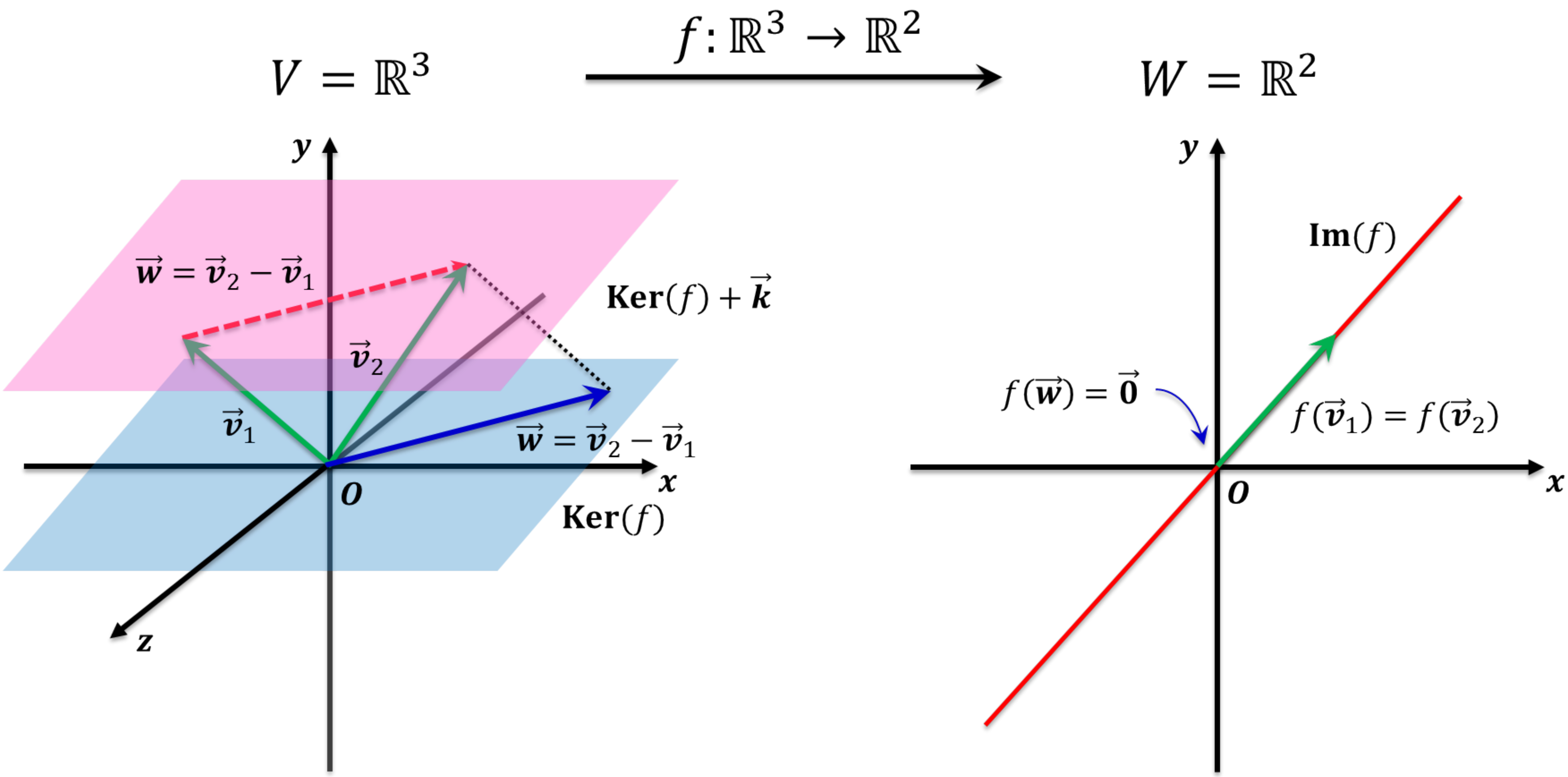
$$\mathrm{Ker}(f) = \{\mathbf{v} \in V \,|\, f(\mathbf{v}) = \mathbf{0}\} \subset V$$
$$\mathrm{Im}(f) = \{f(\mathbf{v}) \,|\, \mathbf{v} \in V\} \subset W$$

Rank of a linear map:

$$\mathrm{rk}(f) = \dim \mathrm{Im}(f)$$

Dimension formula:

$$\dim \mathrm{Ker}(f) + \mathrm{rk}(f) = \dim(V)$$


$V = \mathbb{R}^3$
$f: \mathbb{R}^3 \to \mathbb{R}^2$
$W = \mathbb{R}^2$
y
$\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1$
$\mathbf{Ker}(f) + \vec{\boldsymbol{k}}$
$\vec{\boldsymbol{v}}_2$
$\vec{\boldsymbol{v}}_1$
$\vec{\boldsymbol{w}} = \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1$
x
O
$\mathbf{Ker}(f)$
z
y
$\mathbf{Im}(f)$
$f(\vec{\boldsymbol{w}}) = \vec{\mathbf{0}}$
$f(\vec{\boldsymbol{v}}_1) = f(\vec{\boldsymbol{v}}_2)$
x
O

Basic matrix properties:

- **nxm matrix:** $$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \qquad A_{ij} = a_{ij}$$

- **Row and column vectors:** $$\mathbf{A}_i = \begin{pmatrix} A_{i1} \\ \vdots \\ A_{im} \end{pmatrix}, \qquad \mathbf{A}^j = \begin{pmatrix} A_{1j} \\ \vdots \\ A_{nj} \end{pmatrix}$$

- **Multiplication of matrices and vectors:**

$$A\mathbf{v} = \begin{pmatrix} \mathbf{A}_1 \cdot \mathbf{v} \\ \vdots \\ \mathbf{A}_n \cdot \mathbf{v} \end{pmatrix} \qquad (A\mathbf{v})_i = \sum_{j=1}^{m} A_{ij} v_j$$

<u>Multiplying matrices and vectors</u>

$$A\mathbf{v} = \begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} (1,-2)^T \cdot (-1,5)^T \\ (0,3)^T \cdot (-1,5)^T \\ (5,-1)^T \cdot (-1,5) \end{pmatrix} = \begin{pmatrix} -11 \\ 15 \\ -10 \end{pmatrix}$$

- Lin. maps from matrices: $$\begin{aligned} A : F^m &\to F^n \\ \mathbf{v} &\to A\mathbf{v} \end{aligned}$$

Every linear map $f : F^m \to F^n$ given by a matrix A in this way.

How to find A ? $f(\mathbf{e}_j) = \sum_i A_{ij}\mathbf{e}_i$

<u>Matrix for a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$</u>

$$f(\mathbf{v}) := (\mathbf{a} \cdot \mathbf{v})\mathbf{b} \ , \quad \mathbf{a},\ \mathbf{b} \in \mathbb{R}^3 \text{ fixed}$$

$$f(\mathbf{e}_j) = (\mathbf{a} \cdot \mathbf{e}_j)\mathbf{b} = a_j\mathbf{b} = \sum_{i=1}^{3} b_i a_j \mathbf{e}_i \quad \Longrightarrow \quad A_{ij} = b_i a_j$$

- **Unit matrix:** $\mathbb{1}_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $\qquad (\mathbb{1}_n)_{ij} = \delta_{ij}$

- **Diagonal matrix:** $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} =: \mathrm{diag}(d_1, \ldots, d_n)$

- **Transpose and herm. conjugate:** $(A^T)_{ij} := A_{ji} \qquad A^\dagger := (A^T)^*$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & i \\ 1-2i & -2i \\ 3+i & 4 \end{pmatrix}, \quad B^\dagger = \begin{pmatrix} 1 & 1+2i & 3-i \\ -i & 2i & 4 \end{pmatrix}$$

- **Symmetric and anti-symmetric:** $A = A^T \qquad A = -A^T$

- **Hermitian and anti-hermitian:** $A = A^\dagger \qquad A = -A^\dagger$.

$$A_{\mathrm{symm}} = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \,, \quad A_{\mathrm{anti-symm}} = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$$

$$B_{\mathrm{herm}} = \begin{pmatrix} 2 & 3+4i \\ 3-4i & 1 \end{pmatrix} \,, \quad B_{\mathrm{anti-herm}} = \begin{pmatrix} 2i & 2-5i \\ -2-5i & -i \end{pmatrix}$$

Rank of a matrix:

$$\mathrm{rk}(A) = \dim \mathrm{Span}(\mathbf{A}^1, \cdots, \mathbf{A}^m) = \text{maximal number of lin. indep. column vectors of } A$$

Note: Row rank equals column rank for any matrix.

Rank of a matrix by inspection

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 4 & -3 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$

$\mathbf{A}_1$, $\mathbf{A}_2$ lin. indep. and $\mathbf{A}_3 = \mathbf{A}_1 + \mathbf{A}_2$

$\Longrightarrow \quad \mathrm{rk}(A) = 2$

Matrix multiplication:

$$\overset{r\times m}{C} = \overset{r\times n}{B}\,\overset{n\times m}{A} := \begin{pmatrix} \mathbf{B}_1 \cdot \mathbf{A}^1 & \cdots & \mathbf{B}_1 \cdot \mathbf{A}^m \\ \vdots & & \vdots \\ \mathbf{B}_r \cdot \mathbf{A}^m & \cdots & \mathbf{B}_r \cdot \mathbf{A}^m \end{pmatrix}$$

$$C_{ik} = \sum_{j=1}^{n} B_{ij} A_{jk} = \mathbf{B}_i \cdot \mathbf{A}^k$$

associative, $A(BC) = (AB)C$, but in general not commutative.

$$B = \overset{2\times 3}{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -2 \end{pmatrix}}, \qquad A = \overset{3\times 3}{\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}}$$

$$BA = \overset{2\times 3}{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -2 \end{pmatrix}} \overset{3\times 3}{\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}} = \overset{2\times 3}{\begin{pmatrix} -1 & 2 & 0 \\ 4 & 4 & 3 \end{pmatrix}}$$

Note: $(AB)^T = B^T A^T$

Matrix inverse: For $n \times n$ matrix A

- **defined by** $AA^{-1} = A^{-1}A = \mathbb{1}_n$
- **exists:** A^{-1} exists $\iff$ $\mathrm{rk}(A) = n$
- **properties:** $(AB)^{-1} = B^{-1}A^{-1}$, $((A)^{-1})^{-1} = A$, $(A^T)^{-1} = (A^{-1})^T$

row operations:

(R1) Exchange two rows.
(R2) Add a multiple of one row to another.
(R3) Multiply a row with a non-zero scalar.

upper echelon form:

$$A = \left(\begin{array}{c|cccccc} \cdots & a_{1j_1} & & & & & * \\ \cline{2-3} & & \multicolumn{1}{|c}{a_{2j_2}} & & & \iddots & \\ \cline{3-4} & & & \ddots & & & \\ & & \iddots & & \multicolumn{1}{|c}{a_{rj_r}} & \cdots & \\ \cline{5-7} 0 & & & & & \vdots & \end{array} \right)$$

$\mathrm{rk}(A) = r =$ (number of steps in upper echelon form)

Rank of a matrix by Gaussian elimination

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ 2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 2 & 1 & 0 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2/2} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies \mathrm{rk}(A) = 2$$

Inverse of a matrix by Gaussian elimination

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -2 \\ 1 & -4 & 0 \end{pmatrix} \qquad \mathbb{1}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_3 \to R_3 - R_1 : \quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -2 \\ 0 & -4 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$R_3 \to R_3 + \frac{4}{3} R_2 : \quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \leftarrow \mathrm{rk}(A) = 3 \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & \frac{4}{3} & 1 \end{pmatrix}$$

$$R_2 \to R_2 - 3R_3 : \quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & -3 & -3 \\ -1 & \frac{4}{3} & 1 \end{pmatrix}$$

$$R_1 \to R_1 - 3R_3 : \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \qquad \begin{pmatrix} 4 & -4 & -3 \\ 3 & -3 & -3 \\ -1 & \frac{4}{3} & 1 \end{pmatrix}$$

$$R_2 \to \frac{R_2}{3} : \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \qquad \begin{pmatrix} 4 & -4 & -3 \\ 1 & -1 & -1 \\ -1 & \frac{4}{3} & 1 \end{pmatrix}$$

$$R_3 \to -\frac{3}{2} R_3 : \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_3 \qquad \begin{pmatrix} 4 & -4 & -3 \\ 1 & -1 & -1 \\ \frac{3}{2} & -2 & -\frac{3}{2} \end{pmatrix} = A^{-1}$$

Relation between linear maps and matrices: $f : V \to W$

$$\mathbf{v}_1, \ldots, \mathbf{v}_n \textit{ a basis of } V \textit{ and } \mathbf{w}_1, \ldots, \mathbf{w}_m \textit{ a basis of } W$$

$$f(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i \qquad A = (a_{ij}) \textit{ describes } f$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \begin{aligned} \mathbf{v}_1 &= \mathbf{w}_1 = (1,2)^T \\ \mathbf{v}_2 &= \mathbf{w}_2 = (-1,1)^T \end{aligned}$$

$$B\mathbf{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} = -1\mathbf{v}_1 - 2\mathbf{v}_2 \,, \qquad B\mathbf{v}_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -1\mathbf{v}_1 + 0\mathbf{v}_2$$

$$\Longrightarrow \quad B' = \begin{pmatrix} -1 & -1 \\ -2 & 0 \end{pmatrix} \text{ describes } B \text{ relative to } \{\mathbf{v}_1, \mathbf{v}_2\}$$

Change of basis:

	basis of V	matrix
$f : V \to V$	$\mathbf{v}_1, \ldots, \mathbf{v}_n$	A
	$\mathbf{v}'_1, \ldots, \mathbf{v}'_n$	A'

$$\left.\begin{matrix} A \\ A' \end{matrix}\right\} \quad A' = PAP^{-1} \quad \left\{\begin{aligned} \boldsymbol{\alpha}' &= P\boldsymbol{\alpha} \\ \mathbf{v}_j &= \sum_i P_{ij}\mathbf{v}'_i \end{aligned}\right.$$

3) Systems of linear equations

- m equations in n variables:

$$\overset{m\times n}{\downarrow}\qquad A\mathbf{x}=\mathbf{b} \qquad \text{or} \qquad \begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

- associated homogeneous system:

$$A\mathbf{x}=\mathbf{0} \qquad \text{or} \qquad \begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Interpret matrix A as linear map $A : F^n \to F^m$

$$\text{solutions of hom. system } = \mathrm{Ker}(A)$$
$$\text{with } \dim\mathrm{Ker}(A) = n - \mathrm{rk}(A) \text{ free parameters}$$

$$\text{inhom. system has a solution } \mathbf{x}_0 \Leftrightarrow \mathbf{b} \in \mathrm{Im}(A)$$

$$\text{solution of inhom. system } = \mathbf{x}_0 + \mathrm{Ker}(A)$$

- structure of solution:

(1) $\mathrm{rk}(A) = m$: solution $x_0 + \mathrm{Ker}(A)$

$\dim \mathrm{Ker}(A) = n - m$ free parameters

exists for any $\mathbf{b}$

(2) $\mathrm{rk}(A) < m$

(a) If $\mathbf{b} \in \mathrm{Im}(A)$: solution $x_0 + \mathrm{Ker}(A)$

$\dim \mathrm{Ker}(A) = n - \mathrm{rk}(A)$ free parameters

(b) If $\mathbf{b} \notin \mathrm{Im}(A)$: no solution

Solving lin. eqs. by explicit calculation

$$\begin{aligned} E_1 &: & 2x + 3y - z &= -1 \\ E_2 &: & -x - 2y + z &= 3 \\ E_3 &: & ax + y - 2z &= b\ . \end{aligned} \qquad A = \begin{pmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ a & 1 & -2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ b \end{pmatrix}$$

$$\begin{aligned} E_1 + E_2 &: & x + y &= 2 \\ E_3 + 2E_2 &: & (a-2)x - 3y &= b + 6 \end{aligned} \qquad \Longrightarrow \qquad (a+1)x = b + 12$$

1) $\underline{a \neq -1}$: $x = \dfrac{b+12}{a+1}$, $y = \dfrac{2a - b - 10}{a+1}$, $z = \dfrac{7a - b - 5}{a+1}$ (unique)

2a) $\underline{a = -1 \text{ and } b = -12}$: $x = 2 - y \qquad z = 5 + y$ (line)

2b) $\underline{a = -1 \text{ and } b \neq -12}$: no solution

- **Augmented matrix:** $A' = (A|\mathbf{b})$

- **Criterion for existence of solution:** $\mathbf{b} \in \mathrm{Im}(A) \Longleftrightarrow \mathrm{rk}(A) = \mathrm{rk}(A')$

Solving linear eqs. with row reduction

$$\begin{aligned} x + y - 2z &= 1 \\ 2x - y + 3z &= 0 \\ x - 4y + 9z &= b \end{aligned} \qquad A' = \left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 2 & -1 & 3 & 0 \\ -1 & -4 & 9 & b \end{array}\right)$$

1.) Bring A in A' into upper echelon form:

$$A' \to \left(\begin{array}{rrr|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 7 & -2 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$

Solution exists iff $b = -3$

2.) Set $b = -3$:

$$\left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & -3 & 7 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

3.) Red block $\to$ unit matrix: $A'_{\text{fin}} = \left(\begin{array}{rrr|r} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{7}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ t \end{pmatrix}$

4.) Convert into lin. eqs.:

$$\begin{aligned} x + \frac{1}{3}t &= \frac{1}{3} \\ y - \frac{7}{3}t &= \frac{2}{3} \end{aligned} \qquad \mathbf{x} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{3} \\ \frac{7}{3} \\ 1 \end{pmatrix}$$

5) Determinants

Definition: $\det(\mathbf{a}_1, \cdots, \mathbf{a}_n)$ is

- linear in each argument
- completely anti-symmetric $\quad\Longrightarrow \det(\cdots, \mathbf{a}, \cdots, \mathbf{a}, \cdots) = 0$
- normalized: $\det(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1$

Formula for determinant:

$$\det(A) = \det(\mathbf{A}^1, \cdots, \mathbf{A}^n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} = \epsilon_{i_1 \cdots i_n} a_{i_1 1} \cdots a_{i_n n}$$

$n!$ terms, "pick one entry per column and row"

n=2: $$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \epsilon_{ij} a_i b_j = \epsilon_{12} a_1 b_2 + \epsilon_{21} a_2 b_1 = a_1 b_2 - a_2 b_1$$

n=3: $$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \epsilon_{ijk} a_i b_j c_k = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2$$

$$= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Properties of determinant:

- $\det(A) = \det(A^T)$
- $\det(PAP^{-1}) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = (\det(A))^{-1}$
- A *has an inverse* $\iff \det(A) \neq 0$

Co-factor matrix: $C_{ij} = (-1)^{i+j}\det(A_{(i,j)})$

$$A_{(i,j)} \;:\; A \text{ without row } i \text{ and column } j$$

Expansion by column: $\det(A) = \sum_i (-1)^{i+j} A_{ij}\det(A_{(i,j)})$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix}$$

$$\begin{aligned}\det(A) &= A_{11}\det(A_{(1,1)}) - A_{21}\det(A_{(2,1)}) + A_{31}\det(A_{(3,1)}) \\ &= 2\cdot\det\begin{pmatrix} 2 & -2 \\ 3 & 4 \end{pmatrix} - 1\cdot\det\begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix} + 0\cdot\det\begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} \\ &= 2\cdot 14 - 1\cdot(-4) + 0\cdot 2 = 32\end{aligned}$$

Matrix inverse with determinant: $A^{-1} = \frac{1}{\det(A)} C^T$

Inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \qquad \det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Co-factor method for matrix inverse

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 14 & -4 & 3 \\ 4 & 8 & -6 \\ 2 & 4 & 5 \end{pmatrix} \qquad \det(A) = 32$$

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{32} \begin{pmatrix} 14 & 4 & 2 \\ -4 & 8 & 4 \\ 3 & -6 & 5 \end{pmatrix}$$

Cramer's rule: Solution for $A\mathbf{x} = \mathbf{b}$ where A is $n \times n$, invertible:

$$x_i = \frac{\det(B_{(i)})}{\det(A)} = \frac{\det(\mathbf{A}^1, \cdots, \mathbf{A}^{i-1}, \mathbf{b}, \mathbf{A}^{i+1}, \cdots, \mathbf{A}^n)}{\det(A)}$$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \det(A) = 32$$

$$B_{(1)} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix}, \quad B_{(2)} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & 0 & 4 \end{pmatrix}, \quad B_{(3)} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

$$\det(B_{(1)}) = 22, \qquad \det(B_{(2)}) = 12 \qquad \det(B_{(3)}) = -9$$

$$\Longrightarrow \quad \mathbf{x} = \frac{1}{32}\begin{pmatrix} 22 \\ 12 \\ -9 \end{pmatrix}$$

6) Scalar product

Definition: $\langle \cdot\,, \cdot \rangle : V \times V \to \mathbb{R}(\mathbb{C})$ *satisfying*

$$\textit{(S1)}\ \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle,\ \textit{for a real scalar product},\ F = \mathbb{R}$$

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle^*,\ \textit{for a hermitian scalar product},\ F = \mathbb{C}$$

$$\textit{(S2)}\ \langle \mathbf{v}, \alpha\mathbf{u} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{v}, \mathbf{u} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$$

$$\textit{(S3)}\ \langle \mathbf{v}, \mathbf{v} \rangle > 0\ \textit{if}\ \mathbf{v} \neq \mathbf{0}$$

Norm (length): $|\mathbf{v}| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$

Examples of scalar products

(a) Standard scalar product in $\mathbb{R}^n$

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = \sum_{i=1}^{n} v_i w_i$$

(b) Standard scalar product in $\mathbb{C}^n$

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^\dagger \mathbf{w} = \sum_{i=1}^{n} v_i^* w_i$$

$$|\mathbf{v}|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{n} |v_i|^2$$

(c) real $n \times n$ matrices

$$\langle A, B \rangle := \mathrm{tr}(A^T B) = \sum_{ij} A_{ij} B_{ij}$$

Ortho-normal basis: basis $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n$ of V with $\langle \boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_j \rangle = \delta_{ij}$

Examples of ON basis

(a) $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \implies \mathbf{e}_1, \ldots, \mathbf{e}_n$ ON basis of $\mathbb{R}^n, \mathbb{C}^n$

(b) $\boldsymbol{\epsilon}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\epsilon}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ON basis of $\mathbb{R}^2$ w.r.t. dot product

(c) $\boldsymbol{\nu}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix}, \quad \boldsymbol{\nu}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$ ON basis of $\mathbb{C}^2$, $\boldsymbol{\nu}_i^\dagger \boldsymbol{\nu}_j = \delta_{ij}$

Properties of ortho-normal basis:

- **coordinates:** $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{\epsilon}_i \iff \alpha_i = \langle \boldsymbol{\epsilon}_i, \mathbf{v} \rangle$

Real case: $\mathbf{v} = (2, -3)^T = \alpha_1 \boldsymbol{\epsilon}_1 + \alpha_2 \boldsymbol{\epsilon}_2$

$$\implies \alpha_1 = \boldsymbol{\epsilon}_1 \cdot \mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = -\frac{1}{\sqrt{2}}, \qquad \alpha_2 = \boldsymbol{\epsilon}_2 \cdot \mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{5}{\sqrt{2}}$$

$\underline{\text{Complex case:}}$ $\quad \mathbf{v} = (2,-3)^T = \beta_1 \boldsymbol{\nu}_1 + \beta_2 \boldsymbol{\nu}_2$

$$\Longrightarrow \beta_1 = \boldsymbol{\nu}_1^\dagger \mathbf{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ i \end{pmatrix}^\dagger \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{4+3i}{\sqrt{5}} \,, \quad \beta_2 = \boldsymbol{\nu}_2^\dagger \mathbf{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2i \end{pmatrix}^\dagger \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{2-6i}{\sqrt{5}}$$

- Scalar product:

$$\begin{aligned} \mathbf{v} &= \sum_i \alpha_i \boldsymbol{\epsilon}_i \,, \qquad \alpha_i = \langle \boldsymbol{\epsilon}_i, \mathbf{v} \rangle \\ \mathbf{w} &= \sum_i \beta_i \boldsymbol{\epsilon}_i \,, \qquad \beta_i = \langle \boldsymbol{\epsilon}_i, \mathbf{w} \rangle \end{aligned} \quad \Longrightarrow \quad \langle \mathbf{v}, \mathbf{w} \rangle = \sum_i \alpha_i^* \beta_i = \sum_i \langle \mathbf{v}, \boldsymbol{\epsilon}_i \rangle \langle \boldsymbol{\epsilon}_i, \mathbf{w} \rangle$$

$$\mathbf{v}^\dagger \mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}^\dagger \begin{pmatrix} 2 \\ -3 \end{pmatrix} = 13 \qquad |\beta_1|^2 + |\beta_2|^2 = \left| \frac{4+3i}{\sqrt{5}} \right|^2 + \left| \frac{2-6i}{\sqrt{5}} \right|^2 = 13$$

- Matrix elements: $\quad f(\boldsymbol{\epsilon}_j) = \sum_i A_{ij} \boldsymbol{\epsilon}_i \quad \Longleftrightarrow \quad A_{ij} = \langle \boldsymbol{\epsilon}_i, f(\boldsymbol{\epsilon}_j) \rangle$

Gram-Schmidt procedure: Start with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$\boldsymbol{\epsilon}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}\ , \qquad \mathbf{v}'_k = \mathbf{v}_k - \sum_{i=1}^{k-1} \langle \boldsymbol{\epsilon}_i, \mathbf{v}_k \rangle \boldsymbol{\epsilon}_i\ , \qquad \boldsymbol{\epsilon}_k = \frac{\mathbf{v}'_k}{|\mathbf{v}'_k|}$$

$$\Longrightarrow \boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n \text{ is on ON basis}$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\ , \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\ , \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$\boldsymbol{\epsilon}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{v}'_2 = \mathbf{v}_2 - \langle \boldsymbol{\epsilon}_1, \mathbf{v}_2 \rangle \boldsymbol{\epsilon}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\ , \qquad \boldsymbol{\epsilon}_2 = \frac{\mathbf{v}'_2}{|\mathbf{v}'_2|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}'_3 = \mathbf{v}_3 - \langle \boldsymbol{\epsilon}_1, \mathbf{v}_3 \rangle \boldsymbol{\epsilon}_1 - \langle \boldsymbol{\epsilon}_2, \mathbf{v}_3 \rangle \boldsymbol{\epsilon}_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{7}{6} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}\ , \quad \boldsymbol{\epsilon}_3 = \frac{\mathbf{v}'_3}{|\mathbf{v}'_3|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

Every (finite-dimensional) vector space with scalar product has an ortho-normal basis.

Adjoint map: For $f : V \to V$ adjoint map $f^\dagger : V \to V$ is defined by

$$\langle \mathbf{v}, f\mathbf{w}\rangle = \langle f^\dagger \mathbf{v}, \mathbf{w}\rangle \quad \text{for all } \mathbf{v}, \mathbf{w} \in V$$

$$f \text{ self-adjoint: } f = f^\dagger$$

For ortho-normal basis $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n$, $A_{ij} = \langle \boldsymbol{\epsilon}_i, f(\boldsymbol{\epsilon}_j)\rangle \quad \Longrightarrow \quad \langle \boldsymbol{\epsilon}_i, f^\dagger(\boldsymbol{\epsilon}_j)\rangle = (A^\dagger)_{ij}$

$$f = f^\dagger \quad \Longleftrightarrow \quad A = A^\dagger$$

Orthogonal and unitary maps: $f : V \to V$ satisfying

$$\langle f(\mathbf{v}), f(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle \text{ for all } \mathbf{v}, \mathbf{w} \in V \quad \Longleftrightarrow \quad f^\dagger f = \mathrm{id} \quad \Longleftrightarrow \quad f^{-1} = f^\dagger$$

is called orthogonal (real scalar product) or unitary (complex scalar product)

For ortho-normal basis $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n$, $A_{ij} = \langle \boldsymbol{\epsilon}_i, f(\boldsymbol{\epsilon}_j)\rangle \quad \Longrightarrow$

$$A^\dagger A = \mathbb{1} \quad \Longleftrightarrow \quad A^{-1} = A^\dagger \quad \Longleftrightarrow \quad (\mathbf{A}^i)^\dagger \mathbf{A}^j = \delta_{ij}$$

(Real case: $\dagger \to T$)

Real case: orthogonal matrices

$$A^T A = \mathbb{1} \quad \Longleftrightarrow \quad A^{-1} = A^T \quad \Longleftrightarrow \quad \mathbf{A}^i \cdot \mathbf{A}^j = \delta_{ij} \quad \Longleftrightarrow \quad (A\mathbf{v}) \cdot (A\mathbf{w}) = \mathbf{v} \cdot \mathbf{w} \;\text{ for all } \mathbf{v}, \mathbf{w}$$

$$\Longrightarrow \quad \det(A) = \pm 1$$

$A^T A = \mathbb{1}$ and $\det(A) = +1$: rotations

$A^T A = \mathbb{1}$ and $\det(A) = -1$: rotations combined with reflections

<u>2d rotations</u>

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{rotation by angle } \theta$$

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) \qquad\qquad R(\theta)^{-1} = R(\theta)^T = R(-\theta)$$

3d rotations

$$R_1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \qquad R_2(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \qquad R_3(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

θ_1 around x $\qquad$ θ_2 around y $\qquad$ θ_3 around z

$R(\theta_1, \theta_2, \theta_3) = R_1(\theta_1)R_2(\theta_2)R_3(\theta_3)$ general 3d rotation

Complex case: unitary matrices

$$A^\dagger A = \mathbb{1} \quad \Longleftrightarrow \quad A^{-1} = A^\dagger \quad \Longleftrightarrow \quad (\mathbf{A}^i)^\dagger \mathbf{A}^j = \delta_{ij} \quad \Longleftrightarrow \quad (A\mathbf{v})^\dagger (A\mathbf{w}) = \mathbf{v}^\dagger \mathbf{w} \;\text{ for all } \mathbf{v}, \mathbf{w}$$

$$\Longrightarrow \quad |\det(A)| = 1$$

$A^\dagger A = \mathbb{1}$ and $\det(A) = 1$: special unitary matrices

$A^\dagger A = \mathbb{1}$ and $\det(A) \neq 1$: special unitary matrices times a phase

2d special unitary matrices

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{where} \quad |\alpha|^2 + |\beta|^2 = 1$$

6) Eigenvectors and eigenvalues

Def. of eigenvalues and eigenvectors:

λ eigenvalue of $A : V \to V \quad \Longleftrightarrow \quad$ There is a $\mathbf{v} \neq \mathbf{0}$ with $A\mathbf{v} = \lambda\mathbf{v}$

Then $\mathbf{v}$ is called eigenvector.

Characteristic polynomial:

$$\chi_A(\lambda) := \det(A - \lambda\mathbb{1}) = (-1)^n\lambda^n + (-1)^{n-1}\mathrm{tr}(A)\lambda^{n-1} + \cdots + \det(A)$$

To find eigenvalues: solve $\chi_A(\lambda) = 0$

To find eigenvectors: solve $(A - \lambda\mathbb{1})\mathbf{v} = 0$ for each λ

Key statement:

A can be diagonalized $\quad \Longleftrightarrow \quad$ A has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of eigenvectors

diagonalization: $P = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \qquad P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

Useful statements for hermitian matrices:

- All eigenvalues are real.
- Eigenvectors for different eigenvalues are orthogonal.
- There exists an ortho-normal basis of eigenvectors.

Eigenvectors and eigenvalues for hermitian matrices

Recall: $\langle \mathbf{v}, A\mathbf{w} \rangle = \langle A\mathbf{v}, \mathbf{w} \rangle$ for A hermitian

$$A\mathbf{v} = \lambda \mathbf{v} \quad \Longrightarrow$$

$$\lambda \langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, \lambda \mathbf{v} \rangle = \langle \mathbf{v}, A\mathbf{v} \rangle = \langle A\mathbf{v}, \mathbf{v} \rangle = \langle \lambda \mathbf{v}, \mathbf{v} \rangle = \lambda^* \langle \mathbf{v}, \mathbf{v} \rangle \implies \lambda = \lambda^*$$

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1,\ A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2,\ \lambda_1 \neq \lambda_2 \quad \Longrightarrow$$

$$\lambda_1 \langle \mathbf{v_1}, \mathbf{v}_2 \rangle = \langle \lambda_1 \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle A\mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, A\mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \lambda_2 \mathbf{v}_2 \rangle = \lambda_2 \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$$

$$\Longrightarrow \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$$

Diagonalization of symmetric matrices

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad \chi_A(\lambda) = \det \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} = \lambda(\lambda-1)(\lambda-3)$$

$$\Longrightarrow \quad \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 3$$

$\underline{\lambda_1 = 0}$: $$(A - 0\mathbb{1})\mathbf{v} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ -x+2y-z \\ -y+z \end{pmatrix} \stackrel{!}{=} \mathbf{0} \quad \Longleftrightarrow \quad x = y = z$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}}(1,1,1)^T \text{ so that } \mathbf{v}_1^T\mathbf{v}_1 = 1$$

$\underline{\lambda_2 = 1}$: $$(A - 1\mathbb{1})\mathbf{v} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ -x+y-z \\ -y \end{pmatrix} \stackrel{!}{=} \mathbf{0} \quad \Longleftrightarrow \quad y = 0\,,\ \ x = -z$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}(-1,0,1)^T \text{ so that } \mathbf{v}_2^T\mathbf{v}_2 = 1$$

$\underline{\lambda_3 = 3}$: $$(A - 3\mathbb{1})\mathbf{v} = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x-y \\ -x-y-z \\ -y-2z \end{pmatrix} \stackrel{!}{=} \mathbf{0} \quad \Longleftrightarrow \quad y = -2x\,,\ \ z = x$$

$$\mathbf{v}_3 = \frac{1}{\sqrt{6}}(1,-2,1)^T \text{ so that } \mathbf{v}_3^T\mathbf{v}_3 = 1$$

$$P = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \quad \Longrightarrow \quad P^TAP = \mathrm{diag}(0,1,3)$$

Diagonalization of hermitian matrices

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix} \qquad \chi_A(\lambda) = \begin{pmatrix} 1-\lambda & 2i \\ -2i & 1-\lambda \end{pmatrix} = (\lambda-3)(\lambda+1)$$

$$\Longrightarrow \quad \lambda_1 = 3,\ \lambda_2 = -1$$

$\underline{\lambda_1 = 3:}$

$$(A - 3\mathbb{1})\mathbf{v} = \begin{pmatrix} -2 & 2i \\ -2i & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{!}{=} \mathbf{0} \quad \Longleftrightarrow \quad x = iy$$

$$\Longrightarrow \mathbf{v}_1 = \frac{1}{\sqrt{2}}(i,1)^T \ \text{ so that } \ \mathbf{v}_1^\dagger \mathbf{v}_1 = 1$$

$\underline{\lambda_2 = -1:}$

$$(A + \mathbb{1})\mathbf{v} = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{!}{=} \mathbf{0} \quad \Longleftrightarrow \quad x = -iy$$

$$\Longrightarrow \mathbf{v}_2 = \frac{1}{\sqrt{2}}(-i,1)^T \ \text{ so that } \ \mathbf{v}_2^\dagger \mathbf{v}_2 = 1$$

$$U = (\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \Longrightarrow \quad U^\dagger A U = \mathrm{diag}(3,-1)$$

Note: A, B simultaneously diagonalizable $\Longleftrightarrow$ $[A,B] = 0$

Functions of matrices:

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots \qquad\qquad g(A) = a_0 \mathbb{1}_n + a_1 A + a_2 A^2 + \cdots$$

- **Evaluation by diagonalizing:** $P^{-1}AP = \hat{A} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

$$\Longrightarrow \quad g(A) = \; P\,\mathrm{diag}(g(\lambda_1), \ldots, g(\lambda_n))P^{-1}$$

Function of a matrix by diagonalization

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix} \qquad U = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad U^\dagger A U = \mathrm{diag}(3, -1)$$

$$g(x) = x^n \quad \Longrightarrow \quad g(A) = U\,\mathrm{diag}(3^n, (-1)^n)U^\dagger$$

$$= \frac{1}{2}\begin{pmatrix} (-1)^n + 3^n & -i\,((-1)^n - 3^n) \\ i\,((-1)^n - 3^n) & (-1)^n + 3^n \end{pmatrix}$$

Function of a matrix by explicit evaluation: Pauli matrices

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T \qquad \mathbf{a}\cdot\boldsymbol{\sigma} = a_i \sigma_i$$

$$\sigma_i \sigma_j = \mathbb{1}_2 \delta_{ij} + i\epsilon_{ijk}\sigma_k \qquad \Longrightarrow \qquad (\mathbf{a}\cdot\boldsymbol{\sigma})^{2n} = |\mathbf{a}|^{2n}\mathbb{1}_2 \, , \quad (\mathbf{a}\cdot\boldsymbol{\sigma})^{2n+1} = |\mathbf{a}|^{2n}\mathbf{a}\cdot\boldsymbol{\sigma}$$

$a = i\,\theta\,\mathbf{n}$ with $|\mathbf{n}| = 1$:

$$U := \exp(i\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}(\mathbf{n}\cdot\boldsymbol{\sigma})^n = \cos(\theta)\mathbb{1}_2 + i\sin(\theta)\mathbf{n}\cdot\boldsymbol{\sigma}$$

Quadratic forms: $q(\mathbf{x}) := \sum_{i,j=1}^{n} Q_{ij} x_i x_j = \mathbf{x}^T Q \mathbf{x}$

- **diagonalization:** $Q\mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad P^T = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$

$$PQP^T = \hat{Q} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \qquad \mathbf{y} = P\mathbf{x}$$

$$\Longrightarrow \qquad q(\mathbf{x}) = \mathbf{x}^T P^T \hat{Q} P \mathbf{x} = \mathbf{y}^T \hat{Q} \mathbf{y} = \sum_{i=1}^{n} \lambda_i y_i^2$$

- **quadratic curves and surfaces:**

All points $\mathbf{x}$ satisfying $q(\mathbf{x}) = c\,, \quad c > 0$ constant

all λ_i equal: circle/sphere
all $\lambda_i > 0$: ellipse/ellipsoid
otherwise: hyperbola/hyperboloid

- **for ellipse/ellipsoid:** direction of semi-axes: $\mathbf{v}_i$

length of semi-axes: $\sqrt{\dfrac{c}{\lambda_i}}$

$$q(\mathbf{x}) = x_1^2 - 2x_2x_1 + 2x_2^2 + x_3^2 - 2x_2x_3 = \mathbf{x}^T \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}}_{=A} \mathbf{x} \overset{!}{=} 1$$

$$P = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad P^T A P = \mathrm{diag}(0, 1, 3) \qquad \mathbf{y} = P^T \mathbf{x}$$

$$\Longrightarrow \quad q = y_2^2 + 3y_3^2 \overset{!}{=} 1$$

cylindric in direction $\mathbf{v}_1$ with ellipse cross section in direction $\mathbf{v}_2, \mathbf{v}_3$

length of $\mathbf{v}_2$ half-axis: 1

length of $\mathbf{v}_3$ half-axis: $1/\sqrt{3}$

Good luck!