Vectors and Matrices, Problem Set 4

Eigenvectors, eigenvalues and diagonalization

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Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. Find the eigenvalues and a set of normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} .$$

Verify that its eigenvectors are mutually orthogonal and construct an orthogonal matrix R such that $R^T M R$ is diagonal.

2. Find the eigenvalues and normalized eigenvectors of the Hermitian matrix

$$H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix} .$$

and construct a unitary matrix U such that $U^\dagger H U$ is diagonal. (Remember to use the standard *hermitian* scalar product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^\dagger \mathbf{w}$.)

3. A curve in two-dimensional space is defined by all $\mathbf{x} = (x, y)^T$ which solve the equation $x^2 + 3y^2 - 2xy = 1$.

(a) Show that this equation can be written in the form $\mathbf{x}^T A \mathbf{x} = 1$ and determine the matrix A.

(b) By diagonalizing the matrix A, show that this curve is an ellipse and determine the length of its two axis.

4. (a) Diagonalize the two-dimensional rotation

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

over the complex numbers.

(b) Why can the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

not be diagonalized?

5. A real, symmetric $n \times n$ matrix M is called positive definite if $\mathbf{v}^T M \mathbf{v} > 0$ for all vectors $\mathbf{v}$, negative definite if $-M$ is positive definite and indefinite otherwise.

(a) Show that M is positive definite iff all its eigenvalues are positive and that it is negative definite iff all its eigenvalues are negative.

(b) Focus on the two-dimensional case, $n = 2$, and denote the two eigenvalues of M by λ_1 and λ_2. Why is $\mathrm{tr}(M) = \lambda_1 + \lambda_2$ and $\det(M) = \lambda_1 \lambda_2$?

(c) For $n = 2$, formulate criteria for positive definiteness/negative definiteness/indefiniteness of M in terms of $\mathrm{tr}(M)$ and $\det(M)$.

(d) Are the following matrices positive definite, negative definite or indefinite?

$$M_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} , \quad M_2 = \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix} , \quad M_3 = \begin{pmatrix} -4 & 3 \\ 3 & -5 \end{pmatrix} .$$

6. (a) Show that the characteristic polynomial of an arbitrary 2×2 matrix A can be written as

$$\chi_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) .$$

(b) Do the same for an arbitrary 3×3 matrix A and show that

$$\chi_A(\lambda) = -\lambda^3 + \mathrm{tr}(A)\lambda^2 - \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))\lambda + \det(A) .$$

(c)* For a general $n \times n$ matrix A, what are the coefficients of λ^n, λ^{n-1} and the constant term in the characteristic polynomial $\chi_A(\lambda)$.

7. Consider real, symmetric $n \times n$ matrices P which satisfy $P^2 = P$. (Such matrices are also referred to as projectors.)

(a) Show that the possible eigenvalues of P are either 0 or 1. What does the diagonalized form of P look like?

(b) Explain why such matrices are called projectors and why the dimension of the space projected onto is given by $\mathrm{tr}(P)$.

(c) For an n-dimensional unit vector $\mathbf{n}$ with components n_i let Q be the matrix with components $Q_{ij} = n_i n_j$. Show that Q is a projector and that it projects to a one-dimensional space.

(d) Show that the matrix $P = \mathbf{1} - Q$ with Q defined as in part (c) is also a projector. What is the dimension of the space it projects onto?

8.* (a) Unitary matrices U are characterized by $U^\dagger U = \mathbf{1}$. Show that eigenvalues λ of unitary matrices must be phases, that is, $|\lambda| = 1$.

(b) Consider three-dimensional rotations R. Intuitively, it is clear that R has an eigenvector, $\mathbf{n}$, with eigenvalue 1, which corresponds to the axis of rotation. Construct an argument which proves this fact, that is, show that a three-dimensional real matrix R with $R^T R = \mathbf{1}$ and $\det(R) = 1$ always has an eigenvalue 1.

(c) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$ be an ortho-normal basis of $\mathbb{R}^3$, where $\mathbf{n}$ is the eigenvector with eigenvalue 1 of a rotation R. Work out the matrix which represents the rotation in this basis.

(d) Show that the angle of rotation, φ, of a three-dimensional rotation R satisfies $\cos\varphi = (\mathrm{tr}(R) - 1)/2$.

(e) Show that the matrix

$$R = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & 0 & 3 \\ -1 & -4 & 1 \\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$$

is a rotation. Compute the characteristic polynomial of R and verify that 1 is an eigenvalue. Compute the axis of rotation, $\mathbf{n}$, and $\cos\varphi$, where φ is the angle of rotation.

9.* Consider the system of second order differential equations

$$\frac{d^2\mathbf{x}}{dt^2} = -M\mathbf{x}\,,$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \vdots \\ \frac{d^2x_n}{dt^2} \end{pmatrix},$$

and M is a real, symmetric matrix.

(a) Explain how this system of differential equations can be solved by diagonalizing M.

(b) Discuss the relation between the eigenvalues of M and the qualitative behaviour of the solutions.

(c) Focus on the two-dimensional system

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -kx_1 - lx_2 \\ \frac{d^2x_2}{dt^2} &= -lx_1 - kx_2 \end{aligned}$$

where k and l are arbitrary real constants. Depending on the values of k and l, discuss the qualitative behaviour of the solutions of this system.