Vectors and Matrices, Problem Set 3

Scalar products and determinants

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Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. Calculate the determinants of the matrices

$$(a)\ A = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \quad (b)\ B = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Are the matrices (i) real, (ii) diagonal, (iii) symmetric, (iv) antisymmetric, (v) singular, (vi) orthogonal, (vii) Hermitian, (viii) anti-Hermitian, (ix) unitary?

2. Use the Gram-Schmidt procedure to find an ortho-normal basis of $\mathbb{R}^3$ (with the standard scalar product), starting with the basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

Check your result.

3. Let A be an $n \times n$ matrix.

(a) Why are the statements "A is invertible" and $\det(A) \neq 0$ equivalent?

(b) For which values of the parameters a, b is the matrix

$$A = \begin{pmatrix} a & 1 & a \\ 1 & b & -1 \\ 0 & -1 & a \end{pmatrix}$$

not invertible?

4.* In $\mathbb{R}^n$, we have $n-1$ linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$. Define the vector $\mathbf{w}$ with components $w_i = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{e}_i)$, where $\mathbf{e}_i$ are the standard unit vectors.

(a) Show that $\mathbf{w}$ is perpendicular (with respect to the standard scalar product in $\mathbb{R}^n$) to all vectors $\mathbf{v}_a$, where $a = 1, \ldots, n-1$.

(b) Show that $|\mathbf{w}| = \det(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{n})$, where $\mathbf{n} = \mathbf{w}/|\mathbf{w}|$.

(c) Work out the vector $\mathbf{w}$ if $\mathbf{v}_a = \mathbf{e}_a$, for $a = 1, \ldots, n-1$.

(d) Show that in three dimensions $\mathbf{w}$ can be written in terms of a cross product.

5. Solve the following system of linear equations

$$\begin{aligned} x + 2y + 3z &= 2 \\ 3x + 4y + 5z &= 4 \\ x + 3y + 4z &= 6 \end{aligned}$$

by

(a) calculating the matrix inverse

(b) Cramer's method

(c) row reduction on the augmented matrix.

If you had to write a computer program solving systems of linear equations (of arbitrary and possibly large size) which of the above methods would you base it on?

6.* On the vector space of polynomials $f : \mathbb{R} \to \mathbb{R}$ we define $\langle f,\, g\rangle = \int_{-\infty}^{\infty} dx\, e^{-x^2} f(x)g(x)$.

(a) Why does this define a scalar product?

(b) Consider the polynomials $p_0(x) = 1/n_0$, $p_1(x) = 2x/n_1$ and $p_2(x) = (4x^2 - 2)/n_2$, where the n_a are real numbers. Show that these polynomials are orthogonal under the above scalar product.

(c) Determine the numbers n_a such that the polynomials p_a are normalized to one, so $\langle p_a, p_a\rangle = 1$.

(Hint: You can use that $\int_{-\infty}^{\infty} dx\, x^n e^{-x^2} = 0$ for n odd (why?) and $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$, $\int_{-\infty}^{\infty} dx\, x^2 e^{-x^2} = \sqrt{\pi}/2$, $\int_{-\infty}^{\infty} dx\, x^4 e^{-x^2} = 3\sqrt{\pi}/4$.)

7. The vector space V is equipped with a hermitian scalar product $\langle\cdot,\cdot\rangle$ and an ortho-normal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$.

(a) Show that non-zero and pairwise orthogonal vectors $\mathbf{w}_1, \dots, \mathbf{w}_k$ are linearly independent.

(b) Show that the coordinates v_i of a vector $\mathbf{v}$, relative to the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, are given by $v_i = \langle \mathbf{e}_i, \mathbf{v}\rangle$.

(c) Show that the scalar product of two vectors $\mathbf{u}$, $\mathbf{v}$ can be written as $\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{n} \langle \mathbf{u}, \mathbf{e}_i\rangle\langle \mathbf{e}_i, \mathbf{v}\rangle$.

(d) A second ortho-normal basis $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ is related to the first one by $\mathbf{e}'_j = \sum_i U_{ij}\mathbf{e}_i$, where U_{ij} are complex numbers. Show that $U_{ij} = \langle \mathbf{e}_i, \mathbf{e}'_j\rangle$ and that the matrix U with entries U_{ij} is unitary.

8. Consider $\mathbb{R}^n$ with the standard scalar product $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^T\mathbf{w}$ and orthogonal matrices R, that is, matrices satisfying $\langle R\mathbf{v}, R\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all vectors $\mathbf{v}$, $\mathbf{w}$.

(a) Show that orthogonal matrices can, alternatively, also be characterized by the equation $R^T R = \mathbf{1}$ and that $\det(R) = \pm 1$.

(b) Focus on $n = 2$ and show that two-dimensional orthogonal matrices can be written in the form

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \mp\sin\varphi \\ \sin\varphi & \pm\cos\varphi \end{pmatrix} .$$

Which of these matrices correspond to two-dimensional rotations? What is the interpretation of the other matrices?

(c) For two-dimensional rotations, show that $R(\varphi_1)R(\varphi_2) = R(\varphi_1 + \varphi_2)$.

(d) The vectors $\mathbf{x} = (x, y)^T$ and $\mathbf{x}' = (x', y')^T$ are related by a rotation, so $\mathbf{x}' = R(\varphi)\mathbf{x}$. What is the relation between the two associated complex numbers $z = x + iy$ and $z' = x' + iy'$?

9.* (a) A three-dimensional rotation, R_3, around the z-axis can be constructed from the previous two-dimensional rotations by setting

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and analogously for three-dimensional rotations R_1 and R_2 around the x and y axis. Construct a general three-dimensional rotation by combining three such rotations, that is, work out $R = R_1(\alpha_1)R_2(-\alpha_2)R_3(\alpha_3)$ for three angles α_1, α_2 and α_3.

(b) Show that for small angles α_i the three-dimensional rotations from part (a) can be written as $R = \mathbf{1}_3 + \sum_{i=1}^{3} \alpha_i T_i + \cdots$, where the dots stand for terms of quadratic or higher order in the angles, and determine the (angle-independent) matrices T_1, T_2, T_3.

(c) The change of a vector $\mathbf{x}$ under a rotation R is given by $\delta\mathbf{x} = R\mathbf{x} - \mathbf{x}$. Show that for small-angle rotations this can be written as $\delta\mathbf{x} = \boldsymbol{\alpha} \times \mathbf{x} + \cdots$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$ and the dots stand for quadratic or higher order terms in the angles.

10.* A bi-linear form on $\mathbb{R}^2$ is defined by $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \eta \mathbf{w}$, where $\eta = \mathrm{diag}(-1, 1)$.

(a) Why is this bi-linear form not a scalar product?

(b) Consider the 2×2 matrices Λ which leave the above bi-linear form invariant, that is, which satisfy $\langle \Lambda\mathbf{v}, \Lambda\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all vectors $\mathbf{v}$, $\mathbf{w}$. Show that these matrices can also be characterized by the equation $\Lambda^T \eta \Lambda = \eta$ and that $\det(\Lambda) = \pm 1$.

(c) Show that the matrices Λ with $\det(\Lambda) = 1$ and $\Lambda_{11} > 0$ can be written in the form

$$\Lambda(\xi) = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} ,$$

where ξ is a real parameter. Re-write Λ in terms of the parameter β, defined as $\beta = \tanh\xi$.

(d) Verify that $\Lambda(\xi_1)\Lambda(\xi_2) = \Lambda(\xi_1 + \xi_2)$. What does this rule imply for the parameter β, that is, if $\Lambda(\beta_1)\Lambda(\beta_2) = \Lambda(\beta)$, how does β depend on β_1 and β_2?