Vectors and Matrices, Problem Set 2

Matrices, linear maps and linear equations

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Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. For matrices A, B, C of compatible sizes and scalars α show that

(a) matrix multiplication is associate, so $A(BC) = (AB)C$,

(b) $(A+B)^T = A^T + B^T$, $(\alpha A)^T = \alpha A^T$ and $(AB)^T = B^T A^T$,

(c) $(A+B)^\dagger = A^\dagger + B^\dagger$, $(\alpha A)^\dagger = \alpha^* A^\dagger$ and $(AB)^\dagger = B^\dagger A^\dagger$,

(d) $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^T)^{-1} = (A^{-1})^T$ provided A, B are invertible.

2. Using row operations, work out the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -2 & 1 \\ 3 & 2 & 0 & -4 \\ 1 & -2 & a & 0 \end{pmatrix} ,$$

where a is a real parameter. (Keep in mind that the rank of A depends on the value of a so a case distinction is required.)

3. Using row operations, find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & -3 & 0 \end{pmatrix} .$$

Check your result!

4. A linear map in $\mathbb{R}^2$ is given by the action of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} .$$

(a) Work out the matrix A' which represents this linear map in the basis

$$\mathbf{e}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad \mathbf{e}'_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

(b) Find a 2×2 matrix P such that $A' = PAP^{-1}$.

(c) What is the interpretation of the matrix P?

5.* For a fixed unit vector $\mathbf{n}$ in $\mathbb{R}^3$ define the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\mathbf{v}) = \mathbf{v} - 2(\mathbf{n}\cdot\mathbf{v})\mathbf{n}$.

(a) Show that f is linear and that $f(f(\mathbf{v})) = \mathbf{v}$ for all vectors $\mathbf{v}$.

(b) Work out the matrix A describing f relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of standard unit vectors and show that $A^2 = \mathbf{1}$.

(c) Work out the matrix $\hat{A}$ describing f relative to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$, where $\mathbf{u}_a$, $a = 1, 2$, are two linearly independent vectors perpendicular to $\mathbf{n}$, so $\mathbf{u}_a \cdot \mathbf{n} = 0$.

(d) Discuss your results. What is the geometrical interpretation of f?

6. For a quadractic matrix A, the trace is defined as the sum of its diagonal elements, so $\mathrm{tr}(A) = \sum_i A_{ii}$.

(a) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Show that $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A)$. Discuss this result in relation to basis change of matrices.

(c)* What is the trace of the linear map f defined in the previous question?

7.* Consider the vector space consisting of all quadratic polynomials $p(x) = ax^2 + bx + c$, where a, b, c are arbitrary real coefficients. On this space a map is defined by

$$D(p) = x\frac{d^2p}{dx^2} + (1-x)\frac{dp}{dx} + 2p\,.$$

(a) Why is this map linear?

(b) Work out the matrix, A, which represents D for the standard monomial basis $\{1, x, x^2\}$.

(c) Find the kernel of the matrix A, that is, the vectors $\mathbf{v}$ satisfying $A\mathbf{v} = 0$. Which polynomials p correspond to these vectors?

(d) Show by explicitly applying D that the polynomials p found in part (c) satisfy $D(p) = 0$.

8. Analyze the linear system of equations

$$\begin{aligned} (2-\lambda)x + y + 2z &= 0 \\ x + (4-\lambda)y - z &= 0 \\ 2x - y + (2-\lambda)z &= 0\,, \end{aligned}$$

where λ is an arbitrary real parameter.

(a) What is the rank of the associated coefficient matrix, A, depending on the value of λ?

(b) Based on the rank result in (a) what is your expectation for the qualitative structure of the solution.

(c) Confirm your expectation by an explicit calculation.

9. The linear system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 4z &= \eta \\ x + 4y + 10z &= \eta^2 \end{aligned}$$

depends on the real parameter η.

(a) Show that the rank of the coefficient matrix is two. What does this imply for the qualitative structure of the solution?

(b) Explicitly solve the system for the cases where a solution exists.

10. Solve the linear system

$$\begin{aligned} 3x + 2y - z &= 10 \\ 5x - y - 4z &= 17 \\ x + 5y + \alpha z &= \beta\,, \end{aligned}$$

where α, β are arbitrary real parameters using row reduction on the augmented matrix.

11.* A semi-magic square is a 3×3 matrix of (rational) numbers such that all rows and columns sum up to the same total. A simple example is

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} ,$$

where all rows and columns sum up to 6.

(a) Why do the semi-magic squares form a vector space?

(b) Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} , \quad M_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad M_3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} , \quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} ,$$

form a basis of the semi-magic squares. (Therefore, the dimension of this space is five and all semi-magic squares can be written as linear combinations of the above five matrices.)

12.* Consider an "internet" with n sites, labeled by $i = 1, \ldots, n$. Each site i contains n_i links to some of the other sites and it is linked to by the pages $L_i \subset \{1, \ldots, n\}$. The page rank x_i of each site i is defined by

$$x_i = \sum_{j \in L_i} \frac{x_j}{n_j} . \tag{1}$$

(a) Show that Eq. (1) can be written in matrix form as $A\mathbf{x} = 0$, where $\mathbf{x}$ is the vector with components x_i, and determine the components A_{ij} of the matrix A.

(b) Show a non-trivial solution (that is, a solution $\mathbf{x} \neq 0$) always exists.

(c) Analyze a simple example with $n = 4$ and the following structure of links:

$$\begin{array}{ccc} \mathbf{1} & \rightleftarrows & \mathbf{2} \\ \uparrow & \searrow & \downarrow \\ \mathbf{3} & \longleftarrow & \mathbf{4} \end{array}$$

Write down the page rank equations (1) for this case and solve them. Which page is ranked highest?