Vectors and Matrices, Problem Set 1

Vectors, vector spaces and geometry

Prof Andre Lukas, MT2013

Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. Which of the following sets of vectors are linearly independent? For each linearly dependent set, identify a maximal subset of linearly independent vectors. Provide detailed reasoning in each case.

(a) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (b) $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$ (d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -4 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ 6 \\ -4 \\ 1 \end{pmatrix}$

2. Which of the following sets constitute sub vector spaces of the given vector space? Provide reasoning in each case.

(a) All three-dimensional vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ satisfying $x = y = 2z$.

(b) All two-dimensional vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$.

(c) Within the vector space of real functions $f : \mathbb{R} \to \mathbb{R}$, the subset of even functions, that is, the functions satisfying $f(x) = f(-x)$.

3. Consider the vector space of 3×3 matrices A with real entries A_{ij}. What is the dimension of this vector space?

(a) Show that the subset of symmetric 3×3 matrices, that is, matrices satisfying $A_{ij} = A_{ji}$, forms a sub vector space. Write down an explicit basis for this sub vector space. What is its dimension?

(b) Do the same for the subset of 3×3 anti-symmetric matrices, that is, matrices satisfying $A_{ij} = -A_{ij}$.

(c) Generalize the results from (a) and (b) to $n \times n$ matrices with real entries.

4. Show that the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^3$. Write a general vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of this basis. What are the coordinates of $\mathbf{v}$ relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$?

5.* Consider a real vector space V and two sub vector spaces U and W of V.

(a) Show that the intersection $U \cap W$ is a sub vector space of V.

(b) Show that $U + W$ (the set of all sums $\mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$) is a sub vector space of V.

(c) The dimensions of the above vector spaces are related by $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$. Verify this formula for the specific example where $V = \mathbb{R}^3$, U is spanned by $\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j}$, $\mathbf{u}_2 = \mathbf{k}$ and W is spanned by $\mathbf{w}_1 = \mathbf{j} + \mathbf{k}$, $\mathbf{w}_2 = -\mathbf{i} + 2\mathbf{j}$.

(If you are ambitious, try to prove the dimension formula in (c) in general. Start by writing down a basis for $U \cap W$ and complete this to a basis of U and W, respectively.)

6. For three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ prove the following relations:

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}\,,$

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,,$

(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\,.$

7. For three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, prove the following:

(a) $\mathbf{a} \times \mathbf{b} = \mathbf{a} - \mathbf{b}$ implies that $\mathbf{a} = \mathbf{b}\,,$

(b) $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ implies that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0\,,$

(c) if $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$, this implies that $\mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = \pm|\mathbf{c}| \cdot |\mathbf{a} - \mathbf{b}|\,,$

(d) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})]\,.$

8. The three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form the sides of a triangle.

(a) Show that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{c}| = |\mathbf{c} \times \mathbf{a}|$.

(b) Find the area of the triangle with vertices at $P = (1, 3, 2)$, $Q = (2, -1, 1)$ and $R = (-1, 2, 3)$.

9. (a) Prove that the three vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ are coplanar.

(b) For $\mathbf{a} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}-\mathbf{j}$, find a vector which is coplanar with $\mathbf{a}$ and $\mathbf{b}$, but perpendicular to $\mathbf{a}$.

10.* The three-dimensional vectors $\mathbf{v}_i$, where $i = 1, 2, 3$ are not co-planar with triple product $V = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$. Define the *reciprocal vectors* $\mathbf{v}'_i$ by $\mathbf{v}'_1 = \frac{1}{V}\mathbf{v}_2 \times \mathbf{v}_3$, $\mathbf{v}'_2 = \frac{1}{V}\mathbf{v}_3 \times \mathbf{v}_1$ and $\mathbf{v}'_3 = \frac{1}{V}\mathbf{v}_1 \times \mathbf{v}_2$.

(a) Show that $\mathbf{v}_i \cdot \mathbf{v}'_j = \delta_{ij}$ and, hence, that the coordinates of a vector $\mathbf{w}$ relative to the basis $\mathbf{v}_i$ are given by $\mathbf{w} \cdot \mathbf{v}'_i$.

(b) Show that the triple product $V' = \mathbf{v}'_1 \cdot (\mathbf{v}'_2 \times \mathbf{v}'_3)$ of the reciprocal vectors equals $1/V$.

(c) Show that taking the reciprocal of the reciprocal leads back to the original vectors, that is, show that $\frac{1}{V'}\mathbf{v}'_2 \times \mathbf{v}'_3 = \mathbf{v}_1$, $\frac{1}{V'}\mathbf{v}'_3 \times \mathbf{v}'_1 = \mathbf{v}_2$ and $\frac{1}{V'}\mathbf{v}'_1 \times \mathbf{v}'_2 = \mathbf{v}_3$.

11. What is the shortest distance of $\mathbf{p} = (2, 3, 4)$ from the x-axis?

12. Write down the vector equation of the line

$$\frac{(x-2)}{4} = \frac{(y-1)}{3} = \frac{(z-5)}{2}$$

and find the minimum distance of this line from the origin.

13. Derive an expression for the shortest distance between the two lines
$\mathbf{r}_i = \mathbf{q}_i + \lambda_i\mathbf{m}_i$, where $i = 1, 2$. Hence find the shortest distance between the lines

$$\frac{(x-2)}{2} = (y-3) = \frac{(z+1)}{2} \quad \text{and} \quad (x+2) = \frac{(y+1)}{2} = (z-1)\,.$$

14. Find the Cartesian equation for the plane passing through $P_1 = (2, -1, 1)$, $P_2 = (3, 2, -1)$ and $P_3 = (-1, 3, 2)$.

15. Find the Cartesian and vector description of the plane which contains the three position vectors $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\quad \mathbf{b} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

16. A line goes through the origin and the point $P = (1, 1, 1)$; a plane goes through points $A = (-1, 1, -2)$, $B = (1, 5, -5)$, $C = (0, 2, -3)$. Find the intersection point of the plane and the line.