Mathematical Methods

Problem Sheet 4: "Partial differential equations"

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1) (Laplace equation in two dimensions)

(a) Consider complex coordinates $z = x + iy$ and the 45 degree "cake slice" $\mathcal{V} = \{z \in \mathbb{C} \,|\, \arg(z) \in [0, \pi/4]\,,\ |z| \in [0, a]\}$ in the complex plane. Using complex methods, find solutions ϕ to the two-dimensional Laplace equation on $\mathcal{V}$ which satisfy the boundary conditions $\phi|_{\arg(z)=0} = \phi|_{\arg(z)=\pi/4} = 0$.

(b) From the solutions found in part (a), select the one which, in addition, satisfies the boundary condition $\phi_{|z|=a} = h$, where $h : [0, \pi/4] \to \mathbb{R}$ is a given function.

(c) Show that the most general solution to the two-dimensional Laplace equation in polar coordinates can be written as

$$\phi(r, \varphi) = \frac{a_0}{2} + \frac{\tilde{a}_0}{2} \ln r + \sum_{k=1}^{\infty} (a_k r^k + \tilde{a}_k r^{-k}) \cos(k\varphi) + \sum_{k=1}^{\infty} (b_k r^k + \tilde{b}_k r^{-k}) \sin(k\varphi)\ . \quad (1)$$

(d) Consider the annulus $\mathcal{V} = \{(r, \varphi) \,|\, a_- \le r \le a_+\}$ in the two-dimensional plane and find the solution ϕ to the two-dimensional Laplace equation with boundary conditions $\phi|_{r=a_\pm} = g_\pm$ where $a_+ > a_- > 0$ and $g_\pm : [0, 2\pi] \to \mathbb{R}$ are two given functions.

2) (Laplace equation in three dimensions)

(a) Consider the region $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x} \ge b\}$ outside a ball of radius b and a charge distribution ρ in $\mathcal{V}$ given by a point charge q located at $\mathbf{y} \in \mathcal{V}$. Using the method of mirror charge, solve the Laplace equation $\Delta\phi = -4\pi\rho$ in $\mathcal{V}$, subject to the boundary condition $\phi_{|\mathbf{x}|=b} = 0$. Generalise to the case where ρ corresponds to a number of point charges $q_1, \ldots, q_n$ located at $\mathbf{y}_1, \ldots \mathbf{y}_n \in \mathcal{V}$.

(b) Using properties of the spherical harmonics Y_{lm}, show that the most general solution to the Laplace equation $\Delta\phi = 0$ in spherical coordinates can be written as

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{lm}(\theta, \varphi)\ . \quad (2)$$

For cases where ϕ is independent of φ, write the above expression in terms of Legendre polynomials.

(c) Find the solution to the homogeneous Laplace equation inside a ball $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x} \le a\}$ of radius a, given that the boundary condition is $\phi(a, \theta, \varphi) = \phi_0(1 + \cos\theta)$, where ϕ_0 is a constant.

(d) For the region $\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x} \geq b\}$ outside a ball of radius b find the solution to the homogeneous Laplace equation which satisfies $\phi(b, \theta, \varphi) = \phi_0 \sin^2\theta$ and approaches zero as $r \to \infty$.

3) (Multipole expansion)

(a) Explain why the solution to $\Delta\phi = -4\pi\rho$ which goes to zero at infinity can be written as

$$\phi(\mathbf{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{q_{lm}}{2l+1} \frac{Y_l^m(\mathbf{n})}{r^{l+1}} \,, \qquad q_{lm} = \int_{\mathbb{R}^3} Y_l^m(\mathbf{n}')^* (r')^l \rho(\mathbf{r}')\, d^3r' \,. \tag{3}$$

(b) Find explicitly the spherical harmonics Y_0^0, Y_1^0 and $Y_1^{\pm 1}$.

(c) Use the results from (a) and (b) to write the two lowest terms in the above expansion (that is the monopole term for $l = 0$ and the dipole term for $l = 1$) in terms of Cartesian coordinates. To do this, introduce the total charge $Q = \int_{\mathbb{R}^3} \rho(\mathbf{r}')d^3r'$ and the dipole moment $\mathbf{p} = \int_{\mathbb{R}^3} \mathbf{r}'\rho(\mathbf{r}')d^3r'$.

4) (Strings)
Consider the wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\psi = 0 \,, \tag{4}$$

for the string with $x \in [0, a]$ and Dirichlet boundary conditions $\psi(t, 0) = \psi(t, a) = 0$.

(a) Write down the most general solution for this Dirichlet string using the sine Fourier series.

(b) Find the solution with initial condition $\psi(0, x) = \psi_0 \sin\left(\frac{\pi x}{a}\right)$ and $\dot{\psi}(0, x) = 0$, where ψ_0 is a constant.

(c) Find the solution with initial condition $\psi(0, x) = 0$ and $\dot{\psi}(0, x) = \psi_0 \sin\left(\frac{2\pi x}{a}\right)$.

(d) Find the solution with initial condition $\dot{\psi}(0, x) = 0$ and

$$\psi(0, x) = \begin{cases} \frac{hx}{b} & \text{for} \quad 0 \leq x \leq b \\ \frac{h(a-x)}{a-b} & \text{for} \quad b < x \leq a \end{cases} \tag{5}$$

where b and h are constants. Plot the amplitude b_k/h of the k^{th} eigenmode as a function of $b/a \in [0.1]$ for $k = 1, 2, 3, 4$ and discuss.

5) (Membranes)
Consider the membrane

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}\right)\psi = 0 \,, \tag{6}$$

an the rectangular spatial patch $\mathcal{V} = [0, a] \times [0, b]$ with Dirichlet boundary conditions $\psi(t, 0, y) = \psi(t, a, y) = \psi(t, x, 0) = \psi(t, x, b) = 0$.

(a) Write down the most general solution for the rectangular membrane and find its spectrum of frequencies. What is the ratio of the two lowest eigenfrequencies for the square membrane?

(b) Find the most general solution to a triangular membrane with Dirichlet boundary conditions along all sides, by constraining the general solution of the square membrane along the diagonal $x = y$. What is the spectrum of the triangular membrane?

6) (Eigenfunctions of the Laplacian)
Consider the eigenvalue problem

$$-\Delta\psi = E\psi \tag{7}$$

in three dimensions and with boundary condition $\psi|_{|\mathbf{x}|=a} = 0$. (In quantum mechanics, this correspond to finding the energy eigenstates of a particle in an infinite spherical well.)

(a) Write the three-dimensional Laplacian in terms of a radial coordinate r and the Laplacian Δ_{S^2} on the two-sphere and start by expanding $\psi(r,\theta,\varphi) = \sum_{l.m} R_{l,m}(r)Y_{lm}(\theta,\varphi)$, where R_{lm} are functions to be determined. Show that the R_{lm} must satisfy the differential equation $r^2y'' + 2ry' + (Er^2 - l(l+1))y = 0$.

(b) Introduce a new coordinate $\rho = \sqrt{E}r$ and define $\tilde{y} = \sqrt{\rho}y$. Show that, in terms of these new quantities, the differential equation from part (a) becomes a Bessel differential equation $\rho^2\tilde{y}'' + \rho\tilde{y}' + (\rho^2 - (l+1/2)^2)\tilde{y} = 0$ (where the prime now denotes a ρ derivative).

(c) Find the solutions for R_{lm} and impose the boundary condition $R_{lm}(a) = 0$. Which eigenvalues E do you find?

7 (Heat equation)
Consider the heat equation

$$\kappa\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\psi}{\partial t}\ , \tag{8}$$

where κ is a positive constant, $x \in [0,a]$ and ψ is real-valued. For $t > 0$, we demand Dirichlet boundary conditions $\psi(t,0) = \psi_0$ at $x = 0$ (where ψ_0 is a constant) and von Neumann boundary conditions $\frac{\partial\psi}{\partial x}(t,a) = 0$ at $x = a$. The initial condition is $\psi(0,x) = 0$.

(a) As a preparation, we introduce the functions $g_k : [0,a] \to \mathbb{R}$ defined by

$$g_k(x) = \sqrt{\frac{2}{a}}\sin(q_k x)\ , \qquad q_k = \frac{\pi}{a}\left(k + \frac{1}{2}\right) \tag{9}$$

where $k = 0, 1, \ldots$. Show that these functions satisfy the boundary conditions $g_k(0) = 0$ and $g_k'(a) = 0$, that they form an ortho-normal system relative to the standard scalar product on $L^2([0,a])$ and that they are eigenfunctions of $\frac{d^2}{dx^2}$ with

$$\frac{d^2}{dx^2}g_k = -q_k^2 g_k\ . \tag{10}$$

(b) Argue that the functions g_k from part (a) form, in fact, an ortho-normal basis of the space $L_b^2([0, a])$, of square integrable functions f on $[0, a]$ with a Dirichlet boundary condition $f(0) = 0$ at $x = 0$ and a von Neumann condition $f'(a) = 0$ at $x = a$. (Hint: Think about the sine Fourier series on $[0, 2a]$.)

(c) Based on the results in (a) and (b) argue that the most general ψ with the correct boundary conditions can be written as $\psi(t, x) = \psi_0 + \sum_{k=0}^{\infty} T_k(t) g_k(x)$. Find the solutions for the functions T_k.

(d) Fix the remaining constants in your solution by imposing the initial condition. Compute the average value $\bar{\psi}(t)$ of $\psi(x, t)$ by averaging over $x \in [0, a]$ and find an approximate equation for the time as a function of $r := (\psi_0 - \bar{\psi}(t))/\psi_0$.

Additional computational problems

Computational methods, both numerical and symbolic, are of increasing importance in physics and symbolic computational tools have become significantly more powerful over the past decade or so. This is changing the way physicists work. Much as the introduction of the pocket calculator some 50 years ago has made by-hand numerical calculations unnecessary, modern systems such as Mathematica, can now take over standard symbolic calculations, such as algebraic manipulations or integration. This facilitates powerful checks of by-hand calculations but also allows for calculations which are virtually intractable with a pen-and-paper approach. The following problems present an opportunity to practice some of these methods in the context of topics from the Mathematical Methods course. They are supplementary and voluntary but strongly recommended and hopefully a fun way to engage with symbolic computations early on. The problems are meant for realisation in Mathematica which can be downloaded from the university server. Mathematica is easy to use, has good built-in documentation and many high-level mathematical functions - you can start to experiment immediately.

C1) (Heat equation)
Create an animation for the solution to question 7, above - the heating-up of a rod. Plot the solution $\psi(t, x)$ as a function of x and animate in time t. Also, show the average temperature as a function of t.

C2 (Motion of a string)
Create an animation for the motion of the string from question 4 above.

C3 (Representations of $su(2)$)

(a) Check that the matrices $\tau_i = \sigma_i/2$ (where σ_i are the Pauli matrices) satisfy the standard $su(2)$ commutation relations.

(b) Find the representation matrices T_i of τ_i for a given (but arbitrary) spin j and check that they satisfy the same commutation relations.