Mathematical Methods

Problem Sheet 2: "Fourier series and Fourier integrals"

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1) (Fourier series)

a) Find the Fourier series for the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \leq x < 0 \\ \sin x & \text{for} \quad 0 \leq x \leq \pi \end{cases} \tag{1}$$

(b) Find the Fourier series for the functions $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(x) = x^2$.

(c) Use the result from part (b) to sum the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$.

2) (Sine and cosine Fourier series)

(a) For the functions $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = x \sin x$ find the cosine Fourier series.

(b) For the function from part (a), find the sine Fourier series.

(c) For the function $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = x$ compute the cosine Fourier series.

(d) Compute the sine Fourier series for the function from part (c). Comment on the convergence properties of this series and its cosine Fourier series counterpart. Sum the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

3) (Legendre polynomials as an orthogonal basis)
Denote by $(P_n)_{n=0}^{\infty}$ the Legendre polynomials in $L^2([[-1, 1])$, normalised as $\parallel P_n \parallel^2 = \frac{2}{2n+1}$.

(a) Show that, for any polynomial $p \in L^2([-1, 1])$ with degree less than n, we have $\langle P_n, p \rangle = 0$.

(b) Find explicitly the polynomials $P_0, \ldots, P_4$.

(c) Write the function $f \in L^2([-1, 1])$ with $f(x) = x^4$ as a linear combination of Legendre polynomials.

(d) Write the function $f \in L^2([-1, 1])$ defined by $f(x) = \frac{2}{\sqrt{5-4x}}$ as an expansion in terms of Legendre polynomials. (Hint: Think about the generating function for Legendre polynomials.)

4) (Examples of Fourier transforms)
Recall that the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined as

$$\hat{f}(k) = \mathcal{F}(f)(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx\, f(x) e^{-ikx} \ . \tag{2}$$

(a) Work out the Fourier transform of the function

$$\chi(x) = \begin{cases} 1 & \text{for} \quad |x| \leq 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases} \tag{3}$$

(b) Show that the Fourier transform $\hat{\chi}$ from part (a) is not in $L^1(\mathbb{R})$.

(c) Work out the function $f = \chi \star \chi$, where χ is the function from part (a) and the star denotes the convolution.

(d) Find the Fourier transform of the function f from part (c).

(e) Compute the integral $\int_{\mathbb{R}} dx \, \frac{\sin^4 x}{x^4}$. (Hint: Use the fact that the Fourier transform is unitary.)

5) (Some properties of Fourier transforms)
For a constant $a \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{C}$, define the dilation operator $\mathcal{D}_a$, the translation operator T_a, the modulation operator E_a and the multiplication operator M_g (for a function g) by

$$\mathcal{D}_a(f)(x) := f(ax) \ , \quad T_a(f)(x) := f(x-a) \ , \quad E_a(f)(x) := e^{iax} f(x) \ , \quad M_g(f)(x) := g(x)f(x) \ .$$

Also, denote the Fourier transform by $\mathcal{F}$ and $D_x = \frac{d}{dx}$, $D_k = \frac{d}{dk}$.

(a) Show that $\mathcal{F} \circ T_a = E_{-a} \circ \mathcal{F}$ and $\mathcal{F} \circ E_a = T_a \circ \mathcal{F}$.

(b) Show that $\mathcal{F} \circ \mathcal{D}_a = \frac{1}{|a|} \mathcal{D}_{1/a} \circ \mathcal{F}$.

(c) Show that $\mathcal{F} \circ D_x = M_{ik} \circ \mathcal{F}$ and $\mathcal{F} \circ M_x = i D_k \circ \mathcal{F}$.

6) (More Fourier transforms)

(a) Show that the Gaussian $f(x) = e^{-\frac{x^2}{2}}$ is invariant under Fourier transformation.

(b) Work out the Fourier transform of a Gaussian $f_a(x) = e^{-\frac{x^2}{2a^2}}$ with width a.

(c) Work out the Fourier transform of a Gaussian $f_{a,c}(x) = e^{-\frac{(x-c)^2}{2a^2}}$ with width a and peaked at $x = c$.

7) (Hermite polynomials and Fourier transform)
Denote by $(H_n)_{n=0}^{\infty}$ the standard Hermite polynomials and by $h_n(x) = e^{-\frac{x^2}{2}} H_n(x)$ the Hermite functions which form an orthogonal basis of $L^2(\mathbb{R})$.

(a) Show that the Fourier transform of the function $g(x) = e^{-\frac{x^2}{2}+2xz-z^2}$ is given by $\hat{g}(k) = e^{-\frac{k^2}{2}-2kiz+z^2}$.

(b) Show that the Hermite functions h_n are eigenfunctions of the Fourier transform $\mathcal{F}$, so $\mathcal{F}h_n = \lambda_n h_n$, and compute the eigenvalues λ_n. (Hint: Think about the generating function of the Hermite polynomials and use part (a).)

(c) Every function $f \in L^2(\mathbb{R}^n)$ has an expansion $f = \sum_{n=0}^{\infty} a_n h_n$ in terms of Hermite functions. Write down the expansion of the Fourier transform $\hat{f}$ in terms of Hermite functions and in terms of the coefficients a_n .

Additional computational problems

Computational methods, both numerical and symbolic, are of increasing importance in physics and symbolic computational tools have become significantly more powerful over the past decade or so. This is changing the way physicists work. Much as the introduction of the pocket calculator some 50 years ago has made by-hand numerical calculations unnecessary, modern systems such as Mathematica, can now take over standard symbolic calculations, such as algebraic manipulations or integration. This facilitates powerful checks of by-hand calculations but also allows for calculations which are virtually intractable with a pen-and-paper approach. The following problems present an opportunity to practice some of these methods in the context of topics from the Mathematical Methods course. They are supplementary and voluntary but strongly recommended and hopefully a fun way to engage with symbolic computations early on. The problems are meant for realisation in Mathematica which can be downloaded from the university server. Mathematica is easy to use, has good built-in documentation and many high-level mathematical functions - you can start to experiment immediately.

C1) (Fourier series with Mathematica)
Compute the Fourier series for some of the above functions with Mathematica and check your results by comparing plots of the actual functions with their Fourier series (truncated to some appropriate order). Do this for the following functions:

(a) the function from question 1) a)

(b) the function from question 1) b)

(c) the function from question 2) a)

(d) the function from question 2) b) .

C2) (Fourier transform with Mathematica)
Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = e^{-\gamma t}\sin(\omega_0 t)$ for $t \geq 0$ and by $f(t) = 0$ for $t < 0$, where $\gamma > 0$ and ω_0 are real parameters. Compute its Fourier transform and create an animation which plots f and (the absolute of) its Fourier transform for different values of γ and ω_0.

C3) (Computations with Legendre polynomials)

(a) Assuming that $P_0(x) = 1$ and $P_1(x) = x$ write a short piece of Mathematica code which generates the Legendre polynomials $P_0, P_1, \ldots, P_n$ up to a given n by using the recursion relation. Check your result by verifying that the polynomials you obtain are indeed orthogonal. Plot the first few P_k.

(b) Check that expanding the generating function does indeed lead to the same polynomials.

(c) Expand the function $f(x) = \ln(1 + x^2)$ in terms of Legendre polynomials (with the series truncated at some appropriate finite order). Check your result by plotting and by applying Parseval's equation.