Groups and Representations

Problem Sheet 2

Deadline: Fri, week 5, noon

1) (Lie-groups and their Lie-algebras)

a) Derive the Lie-algebras of $SO(4)$ and $SU(2) \times SU(2)$ and show that they are isomorphic. [6]

b) Do the same for $SO(6)$ and $SU(4)$ (Hint: It is helpful to contruct a basis for the $SU(4)$ Lie algebra starting with gamma matrices in six Euklidean dimensions - these are 8×8 matrices - and their antisymmetrized products.) [8]

c) Show that the $2n \times 2n$ real matrices M satisfying $M^T \eta M = \eta$ where

$$\eta = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

form a group. This group is called the symplectic group $Sp(2n)$. Find the Lie-algebra $sp(2n)$ of $Sp(2n)$ and its Cartan subalgebra. Further, determine $\dim(sp(2n))$ and $\mathrm{rk}(sp(2n))$.[6]

2) (Relation between $SU(n) \times U(1)$ and $U(n)$) A map $f : SU(n) \times U(1) \to U(n)$ is defined by $f((U, z)) = zU$, where $U \in SU(n)$ and $z \in U(1)$.

a) Show that this map f defines a group homomorphism. [5]

b) Work out $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$. [10]

c) From the result in b), deduce the relation between $SU(n) \times U(1)$ and $U(n)$. [5]

3) (The Lorentz group) A Dirac spinor ψ transforms in the representation $R_D = (1/2, 0) \oplus (0, 1/2)$ of the Lorentz group and can be written as

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

where χ_L and χ_R are left- and right-handed Weyl spinors. The representation matrices $R_D(M)$ acting on ψ are given by

$$R_D(M) = \begin{pmatrix} R_L(M) & 0 \\ 0 & R_R(M) \end{pmatrix} .$$

Define the gamma matrices γ_μ by

$$\gamma_0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} , \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} .$$

a) Using the explicit expressions for $R_L(M)$ and $R_R(M)$, show that an infinitesimal transformation of ψ takes the form $\delta\psi = i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi$ where $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and $\epsilon^{\mu\nu}$ are small parameters. [5]

b) Show explicitly that the matrices $\sigma_{\mu\nu}$ form a representation of the Lorentz group Lie algebra. [5]

c) Use the relation between the Lorentz group and $SL(2,C)$ to show that $R_D(M)^{-1}\gamma_\mu R_D(M) = R_V(M)_\mu{}^\nu \gamma_\nu$. [6]

d) Proof that the Dirac equation for the spinor ψ with mass m is Lorentz-covariant by applying the result c). [4]

4) ($SU(5)$, tensor methods and branching)

a) Find the Young-tableaux and associated tensors for the representations $\mathbf{1}$, $\mathbf{5}$, $\bar{\mathbf{5}}$, $\mathbf{10}$, $\mathbf{15}$ and $\mathbf{24}$ of $SU(5)$. [4]

b) Show that

$$\begin{aligned}
\mathbf{5} \times \bar{\mathbf{5}} &= \mathbf{1} + \mathbf{24} \\
\mathbf{5} \times \mathbf{5} &= \mathbf{10} + \mathbf{15} \\
\bar{\mathbf{5}} \times \mathbf{10} &= \mathbf{5} + \mathbf{45} \\
\mathbf{10} \times \mathbf{10} &= \bar{\mathbf{5}} + \overline{\mathbf{45}} + \overline{\mathbf{50}}
\end{aligned}$$

using Young-tableaux. [4]

c) Using the obvious embedding of $SU(3) \times SU(2)$ into $SU(5)$, such that, $U_3 \in SU(3)$ and $U_2 \in SU(2)$ are embedded as

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & U_2 \end{pmatrix} \in SU(5) \,,$$

work out the branching of the representations $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\mathbf{10}$ under $SU(3) \times SU(2)$. [4]

d) Show that the unique $U(1)$ sub-group of $SU(5)$ which commutes with $SU(3) \times SU(2)$ (embedded into $SU(5)$ as above) is given by the matrices $\mathrm{diag}(e^{-2i\alpha}, e^{-2i\alpha}, e^{-2i\alpha}, e^{3i\alpha}, e^{3i\alpha})$, where $\alpha \in \mathbb{R}$. For this $U(1)$, work out the charges for all $SU(3) \times SU(2)$ representations which appear in the branchings worked out in c). [4]

e) Using tensor methods, write down the $SU(5)$ singlet in $\mathbf{5} \otimes \mathbf{10} \otimes \mathbf{10}$ and $\bar{\mathbf{5}} \otimes \bar{\mathbf{5}} \otimes \mathbf{10}$. Using the branchings from c), write these singlets in terms of $SU(3) \times SU(2)$ representations. [4]

5) (Relation between $SU(n)$ and $SO(2n)$) Write matrices $U \in SU(n)$ as $U = U_R + iU_I$, where $U_R = \mathrm{Re}(U)$ and $U_I = \mathrm{Im}(U)$ and define the map $f : SU(n) \to Gl(\mathbb{R}^{2n})$ by

$$f(U) = \begin{pmatrix} U_R & -U_I \\ U_I & U_R \end{pmatrix} \,.$$

a) Show that $\mathrm{Im}(f) \subset SO(2n)$. [5]

b) Show that f is an injective group homomorphism. (Hence, it defines an embedding of $SU(n)$ into $SO(2n)$ and we can think of $SU(n)$ as a sub-group of $SO(2n)$.) [5]

c) Show that the branching of the fundamental representation, $\mathbf{2n}$, of $SO(2n)$ under the $SU(5)$ sub-group in b) is given by $\mathbf{2n} \to \mathbf{n} \oplus \bar{\mathbf{n}}$, where $\mathbf{n}$ is the fundamental of $SU(n)$. [5]

d) For $n = 5$, work out the branching of the adjoint of $SO(10)$ under $SU(5)$. [5]