Groups and Representations

MT 2020

Lecture 1

$$\begin{tikzcd}
\text{physics} & \text{mathematics} \\
\text{symmetry} \arrow[d, "\text{“acts on”}"'] \arrow[r, phantom, "\cong"] & \text{group} \arrow[d, "\text{(linear) representation}"] \\
\text{“building blocks”} \arrow[r, phantom, "\cong"] & \text{vector spaces}
\end{tikzcd}$$

V vector space, group G

$GL(V)$: general linear group

representation of G : $R : G \to GL(V)$

such that $R(g_1 \cdot g_2) = R(g_1) R(g_2)$

$g \longmapsto R(g)$

$$\begin{array}{c} v \\ \cap \\ V \end{array} \longmapsto R(g) v$$

Basics of groups and representations

Groups

<u>Def</u> (<u>groups</u>) A <u>group</u> G is a set with a map $\circ : G \times G \to G$, called multiplication, such that

(i) $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \quad \forall g_1, g_2, g_3 \in G$ "associativ[e]"

(ii) $\exists e \in G : e \circ g = g \quad \forall g \in G$ "neutral element"

(iii) $\forall g \in G \ \exists g' \in G : g' \cdot g = e$ "inverse"

If $g_1 \cdot g_2 = g_2 \cdot g_1 \ \forall g_1, g_2 \in G$ the group is called <u>Abelian</u>

<u>Remark:</u> The left-inverse is unique and is also the right-inverse. The left unit is unique and is also the right unit. $g' = g^{-1}$

$$\Rightarrow (g^{-1})^{-1} = g \ , \ (g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1}$$

both inverse to g^{-1} both inverse of $g_1 \cdot g_2$

Reminder: Equivalence relations

Def: (Equivalence relation) A relation $\sim$ on a set S is an <u>equivalence relation</u> if

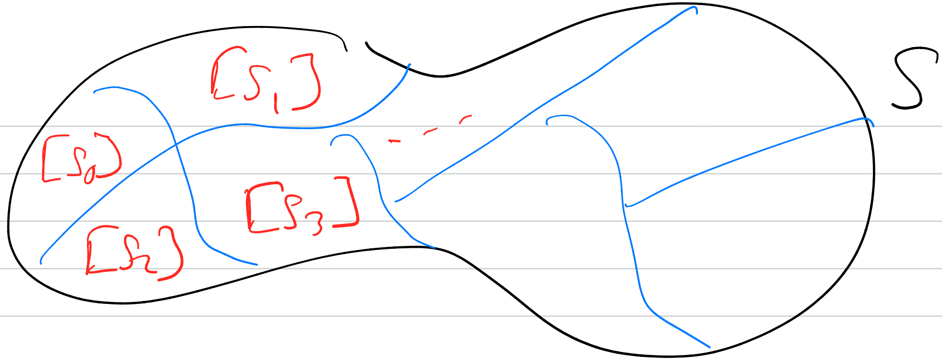
(i) $s \sim s \quad \forall s \in S$ — reflexivity

(ii) $s \sim s' \Rightarrow s' \sim s, \quad \forall s, s' \in S$ — symmetry

(iii) $s \sim s'$ and $s' \sim s'' \Rightarrow s \sim s'' \quad \forall s, s', s'' \in S$ — transitivity

The <u>equivalence class</u> $[s]$ of $s \in S$ is defined as $[s] = \{ s' \in S \mid s' \sim s \}$.

<u>Prop:</u> Two equivalence classes are either disjoint or equal.



Def: $g_1, g_2 \in G$ are called <u>conjugate</u> if $\exists g \in G : g_2 = g g_1 g^{-1}$.

This is an equivalence relation. The equivalence classes called <u>conjugacy classes</u>.

Def: (subgroup) A <u>subgroup H</u> of G is a non-empty subset $H \subset G$ which forms

a group under the multiplication G.
trivial subgroups: $\{e\} \subset G$, $G \subset G$
other subgroups: non-trivial.

$$H \subset G \text{ subgroup} \iff \begin{cases} (i) \ e \in H \\ (ii) \ h \in H \Rightarrow h^{-1} \in H \\ (iii) \ h_1, h_2 \in H \Rightarrow h_1 \cdot h_2 \in H \end{cases}$$

For $H \subset G$ sub-group we can define the equiv. relation

$$g_1 \sim g_2 \iff g_1^{-1} \cdot g_2 \in H \qquad (*)$$

<u>Def</u>: (Cosets) The equiv. classes for $(*)$ are called <u>(left) cosets</u> of H in G

$gH = \{gh \mid h \in H\}$.

For finite groups (finite number of elements $|G|$):

- every coset has the same number of elements (the same as H)
- $|G| = k|H|$ where $k \in \mathbb{N}$
- If $|G|$ is prime, then G has no proper sub-group

<u>Remark</u> for finite group G: $e = g^0, g^1, g^2, \dots$

$\exists k > l: g^k = g^l \Rightarrow g^{k-l} = e$

The smallest $p \in \mathbb{N}$ for which $g^p = e$ is called the <u>order</u> of g.

Remark: We can also define right cosets.

Def: (Normal subgroups) A sub-group $H \subset G$ is called normal if $gH = Hg \ \forall g \in G$ (left and right cosets are the same)

$H \subset G$ normal, cosets $G/H = \{gH \mid g \in G\}$ "quotient space" forms a group with multiplication

$$(g_1 H)(g_2 H) = (g_1 g_2)H$$

neutral element of G/H: $eH = H$

inverse of gH: $g^{-1}H$

$$\underbrace{(g_1 h_1 H)}_{= g_1 H}\underbrace{(g_2 h_2 H)}_{g_2 H} = (g_1 h_1 g_2 h_2) H$$
$$= (g_1 h_1 g_2) H$$
$$= g_1 h_1 H g_2 = g_1 H g_2$$
$$= g_2 g_2 H$$

Def: (Homomorphisms) A map $f: G \to \tilde{G}$ is called a (group) <u>homomorphism</u> if

$$f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2) \quad \forall g_1, g_2 \in G.$$

$$\mathrm{Ker}(f) = \{ g \in G \mid f(g) = \tilde{e} \} \subset G \quad \text{"kernel"}$$
$$\mathrm{Im}(f) = \{ f(g) \mid g \in G \} \subset \tilde{G} \quad \text{"image"}$$

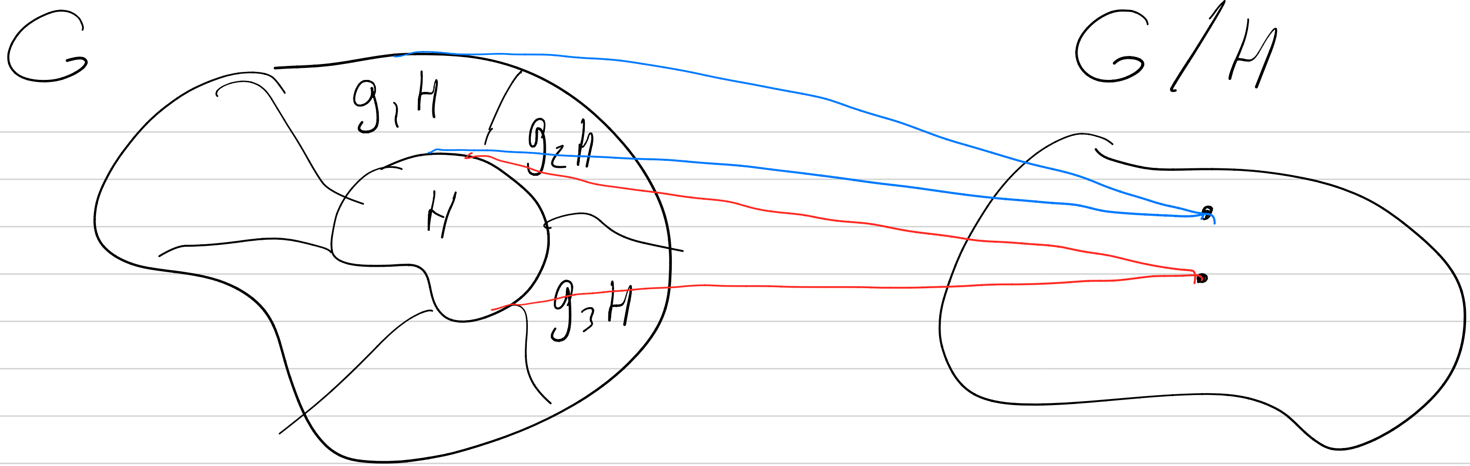
Remarks

- $f(e) = \tilde{e}$ $\Rightarrow e \in \ker(f)$, $f(g^{-1}) = f(g)^{-1}$
- $\ker(f)$ is a normal subgroup of G
- $\mathrm{Im}(f)$ is a subgroup of $\tilde{G}$
- f one-to-one (injective) $\iff \ker(f) = \{e\}$
- f is onto (surjective) $\iff \mathrm{Im}(f) = \tilde{G}$

Theorem: (Isomorphism theorem) $f: G \to \tilde{G}$ homomorphism, then

$$G/\ker(f) \cong \mathrm{Im}(f)$$

$\exists$ bij. group hom.



Representations

Basics

vector space V over $\mathbb{F}$ $(=\mathbb{R}$ or $=\mathbb{C})$ general linear group $GL(V) = \mathrm{Aut}(V) = \{f : V \to V \mid f$ linear + bijective$\}$

$GL(\mathbb{R}^n) = \{ n \times n \text{ invertible matrices with real entries} \}$

$GL(\mathbb{C}^n) = \{ n \times n \text{ invertible matrices with complex entries} \}$

$GL(V)$ forms a group

$GL(V)$ acts on V : $v \mapsto f(v)$

Def: (representation) A <u>representation</u> R of G is a group homomorphism $R : G \to GL(V)$ (V is vector space over $\mathbb{F}$). The <u>dimension</u> of R is defined as $\dim(R) = \dim_{\mathbb{F}}(V)$.

<u>Remarks</u> • $V = \mathbb{R}^n$ or $\mathbb{C}^n$, $R(\overset{\in G}{g})$ is an $n \times n$ matrix

- $R(g_1 \cdot g_2) = R(g_1)\, R(g_2)$
- $R(e) = \mathrm{id}_V$, $R(g^{-1}) = R(g)^{-1}$

$$GL(V) \subset End(V) = \{ f : V \to V \,/\, f \text{ linear} \}$$

group   vector space

Def: (equivalent representations) Two repr. $R_1 : G \to GL(V_1)$ and $R_2 : G \to GL(V_2)$ are called <u>equivalent</u> if there $\exists$ an isomorphism $\phi : V_1 \to V_2$ s.t.

$$R_1(g) = \phi^{-1} \circ R_2(g) \circ \phi \quad \forall g \in G.$$

In this case we write $R_1 \cong R_2$, or else, $R_2 \not\cong R_2$.

If $R_1 \cong R_2 \Rightarrow \dim(R_1) = \dim(R_2)$

Often, there is additional structure: scalar product $\langle \cdot, \cdot \rangle$ on V $\rightsquigarrow$ V inner product space

unitary group $\mathcal{U}(V) = \{ f \in GL(V) \mid \langle f(v), f(w) \rangle$
$= \langle v, w \rangle \ \forall v, w \in V \}$
$= \{ f \in GL(V) \mid f^{+} \circ f = id_V \}$

$$\left.\begin{array}{l} \mathcal{U}(\mathbb{R}^n) = \{ n \times n \text{ orthogonal matrices} \} = O(n) \\ \mathcal{U}(\mathbb{C}^n) = \{ n \times n \text{ unitary matrices} \} = \mathcal{U}(n) \end{array}\right\}$$

$\mathcal{U}(V) \subset GL(V)$ subgroup

Def: (unitary repr.) A repr. $R : G \to GL(V)$ is called unitary if $\mathrm{Im}(R) \subset \mathcal{U}(V)$.

Def: (Faithful repr.) A repr. $R \to GL(V)$ is called faithful if it is one-to-one (iff $\mathrm{Ker}(R) = \{e\}$).

Def: (reducible and irreducible reps). A repr. $R : G \to GL(V)$ is called reducible if there $\exists$ a non-trivial subspace $U \subset V$ (i.e. $U \neq \{0\}, V$) s.t. $R(g)U \subset U$, $\forall g \in G$. Otherwise, R is called irreducible or an irrep basis of U, complete to a basis of V

$$[R(g)] = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} \begin{array}{l} \text{basis of } U \\ \text{other basis vectors} \end{array}$$

$$= \begin{pmatrix} A(g)u \\ 0 \cdot u \end{pmatrix}$$

<u>Def</u>: (Fully reducible repr.) $R: G \to GL(V)$ repr.
If $V = U_1 \oplus \cdots \oplus U_k$, such that
$R(g)U_i \subset U_i \;\; \forall g \in G$, $i = 1, \ldots, k$ and
the repr. $R_i : G \to GL(U_i)$ defined by
$R_i(g) = R(g)|_{U_i}$ are irreducible, the R
is called <u>fully reducible</u>. In this case

we write $R = R_1 \oplus \cdots \oplus R_k$

If R is fully red.:

$$[R(g)] = \begin{pmatrix} [R_1(g)] & & 0 \\ & \ddots & \\ 0 & & [R_k(g)] \end{pmatrix} \begin{array}{c} U_1 \\ \vdots \\ U_k \end{array}$$

$$R(g_1)R(g_2) = \begin{pmatrix} R_1(g_1) & & \\ & \ddots & \\ & & R_k(g_1) \end{pmatrix} \begin{pmatrix} R_1(g_2) & & \\ & \ddots & \\ & & R_k(g_2) \end{pmatrix}$$

$$= \begin{pmatrix} R_1(g_1)R_1(g_2) & & \\ & \ddots & \\ & & R_k(g_1)R_k(g_2) \end{pmatrix}$$

$$\Big\downarrow \qquad \quad ||$$

$$R(g_1 \circ g_2) = \begin{pmatrix} R_1(g_1 g_2) & & \\ & \ddots & \\ & & R_k(g_1 g_2) \end{pmatrix}$$

$\Rightarrow R_i(g_1 g_2) = R_i(g_1) R_i(g_2)$

If $R = R_1 \oplus \cdots \oplus R_k \Rightarrow \dim(R) = \sum_{i=1}^{k} \dim(R_i)$

Prop: Unitary repr. are fully reducible.

Proof: $R : G \to U(V)$ on V, inner product space with scalar product $\langle \cdot , \cdot \rangle$. If R is an irrep then its fully reducible.

Assume R is reducible, $R(g) U \subset U$.

$W = U^{\perp} = \{ w \in V \mid \langle w, u \rangle = 0 \ \forall u \in U \}$

$\Rightarrow V = U \oplus W$

$w \in W$, $u \in U$: $\langle \overset{\swarrow \text{unitary}}{R(g)} w, u \rangle = \langle w, \underbrace{R(g)^{-1} u}_{\in U} \rangle$

$\Rightarrow R(g) w \in W$ or $= 0$

$R(g) W \subset W$

If $R|_U$, $R|_W$ are repr. If they are irr. we are finished. Otherwise, iterate. $\square$

Cor: Any repr. of a finite group is unitary and, hence, fully reducible.

Proof: $R: G \to GL(V)$, introduce scalar product $\langle \cdot, \cdot \rangle_0$ on V. Define a new scalar product

$$\langle v, w \rangle = \sum_{g' \in G} \langle R(g')v, R(g')w \rangle_0$$

$$\Rightarrow \langle R(g)v, R(g)w \rangle = \langle v, w \rangle \quad \forall v, w \in V$$

$\Rightarrow$ R is unitary relative to $\langle \cdot , \cdot \rangle$ $\square$

$$\langle R(g)v, R(g)w \rangle = \sum_{g' \in G} \langle R(g')R(g)v, R(g')R(g)w \rangle_0$$

$$= \sum_{g' \in G} \langle R(g'g)v, R(g'g)w \rangle_0$$

$$R(g): V \to V \qquad\qquad V = U_1 \oplus \cdots \oplus U_k$$

$$R(g)\big|_{U_i}: U_i \to U_i \qquad\qquad R(g)\, U_i \subset U_i$$

$$\left.\begin{array}{l} R: G \to GL(V) \\ R(g): V \to V \end{array}\right\}$$

$$\begin{array}{c} R(g) \\ \cap \\ GL(V) \end{array}$$