Groups and Representations

MMathPhys/MScMTP lecture course

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MT 2020

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Preface

Symmetries have become a key idea in modern physics and they are an indispensable tool for the construction of new physical theories. They play an important role in the formulation of practically all established physical theories, from Classical/Relativistic Mechanics, Electrodynamics, Quantum Mechanics, General Relativity to the Standard Model of Particle Physics.

The word "symmetry" used in a physics context (usually) refers to the mathematical structure of a **group**, so this is what we will have to study. In physics, the typical problem is to construct a theory which "behaves" in a certain defined way under the action of a symmetry, for example, which is invariant. To tackle such a problem we need to know how symmetries act on the basic building blocks of physical theories and these building blocks are often elements of vector spaces. (Think, for example, of the trajectory $\mathbf{r}(t)$ of a particle in Classical Mechanics which, for every time t, is an element of $\mathbb{R}^3$, a four-vector $x^\mu(t)$ which is an element of $\mathbb{R}^4$ or the electric and magnetic fields which, at each point in space-time, are elements of $\mathbb{R}^3$.) Hence, we need to study the action of groups on vector spaces and the mathematical theory dealing with this problem is called (linear) **representation theory of groups**. The translation between physical and mathematical terminology is summarised in the diagram below.

$$\begin{array}{ccccc} & \textbf{physics} & & \textbf{mathematics} & \\ & \text{symmetry} & \cong & \text{group} & \\ \text{action on} & \downarrow & & \downarrow & \text{representation} \\ & \text{building blocks} & \cong & \text{vector spaces} & \end{array}$$

Groups and their representations form a large area of mathematics and a comprehensive treatment can easily fill two or three lecture courses. Our selection of material is guided by a desire of at least a certain degree of mathematical consistency and rigour on the one hand and by what is important for a theoretical physicist on the other hand. We have largely refrained from presenting physicists' treatments when it comes to developing the basic structure. The subject has become such an established part of modern physics, that following the standard mathematical route seems appropriate.

In the first chapter we review some basics of group theory and introduce the key idea of linear representations. The first application of this idea is to **finite groups**, using the technique of group characters. The remainder of these notes deals with **Lie groups** and **Lie algebras** and their representations. After developing the basics, we discuss some of the most important examples - orthogonal and unitary groups, the Lorentz group - in detail. The notes end with the classification of semi-simple Lie algebras via **Dynkin diagrams** and the **Dynkin formalism** to describe linear representations of these algebras.

Acknowledgements I would like to thank the 2020/21 MMathPhys/MScMTP class for asking many interesting questions and for pointing out mistakes and misprints. Special thanks go to Wing Fung Leung whose careful reading helped to remove a significant number of errors in the manuscript. Another special thanks to Lewis Chan from the 2021/22 MMathPhys/MScMTP class for correcting many misprints.

Chapter 1

Basics of groups and representations

The first part of this chapter is a lightning review of some basic group properties. For those who have had a basic course on groups before this is nothing new. In this case, skip it or consider it as a reminder. Then we introduce linear representations of groups and develop some of their elementary properties. The chapter ends with two examples of the simplest Abelian groups, $\mathbb{Z}_n$ and $U(1)$, and their representations.

1.1 Groups

Definition of group

We begin with the general definition of a group.

Definition 1.1. *(Groups) A* ***group*** *G is a set together with a map $\cdot : G \times G \to G$ called multiplication (in particular, G is closed under multiplication) which satisfies*

(i)	$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \quad \forall\, g_1, g_2, g_3 \in G$	*(associativity)*
(ii)	$\exists\, e \in G \,:\, e \cdot g = g \quad \forall\, g \in G$	*(neutral or unit element)*
(iii)	$\forall\, g \in G,\ \exists\, g' \in G \,:\, g' \cdot g = e$	*(left inverse)*

The group is called ***Abelian*** *if $g_1 \cdot g_2 = g_2 \cdot g_1\ \forall\, g_1, g_2 \in G$.*

This definition looks somewhat asymmetrical since the unit and the inverse are only postulated when acting from the left. However, this is not a problem due to the following remark.

Remark 1.1. *The left inverse is unique (and is in the following denoted by $g' = g^{-1}$) and is also the right inverse. The left unit element is unique and is also the right unit.*

This implies that $(g^{-1})^{-1} = g$ and $(g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1}$

Exercise 1.1. *Prove the statements in the previous remark (start by showing that the left-inverse is also the right-inverse).*

Note that the group multiplication is not in general commutative and many of our examples will be non-Abelian groups. From now on we will usually drop the symbol $\cdot$ to indicate group multiplication and simply write $g_1 \cdot g_2 = g_1 g_2$.

A simple method to obtain new groups from given ones is the **direct product** construction. For two groups G_1 and G_2 with neutral elements e_1 and e_2 the Cartesian product $G_1 \times G_2$ can be made into a group, called the **direct product group** of G_1 and G_2, by defining the multiplication component-wise, so $(g_1, g_2) \cdot (\tilde{g}_1, \tilde{g}_2) = (g_1 \cdot \tilde{g}_1, g_2 \cdot \tilde{g}_2)$. The neutral element of

$G_1 \times G_2$ is (e_1, e_2) and the inverse for (g_1, g_2) is (g_1^{-1}, g_2^{-1}).

Reminder: Equivalence relations

The following is a basic reminder of equivalence classes and equivalence relations, a structure which is required for our subsequent discussion of groups.

Definition 1.2. *(Equivalence relation) A relation $\sim$ on a set S is an **equivalence relation** iff*

(i)	$s \sim s \quad \forall\, s \in S$	*(reflexivity)*
(ii)	$s \sim s' \implies s' \sim s \quad \forall\, s, s' \in S$	*(symmetry)*
(iii)	$s \sim s'$ and $s' \sim s'' \implies s \sim s'' \quad \forall\, s, s', s'' \in S$	*(transitivity)*

*The **equivalence class** $[s]$ of some $s \in S$ is defined as $[s] = \{s' \in S \mid s' \sim s\}$*

Proposition 1.1. *Two equivalence classes are either disjoint or equal.*

Proof. If the two equivalence classes $[s_1]$ and $[s_2]$ are disjoint we are finished so assume that $s \in [s_1] \cap [s_2]$. This implies that $s \sim s_1 \sim s_2$. Start with any $\tilde{s} \in [s_1]$. Then transitivity implies that $\tilde{s} \sim s_1 \sim s_2$ and, hence, $\tilde{s} \in [s_2]$. It follows that $[s_1] \subset [s_2]$ and the reverse inclusion follows from the same argument with the roles of $[s_1]$ and $[s_2]$ exchanged. $\square$

An equivalence class consists of all elements of S which are related to one another and the set S "cleanly" splits up into disjoint equivalence classes (see Fig. 1.1). The set of all equivalence classes is also denoted by $S/\sim$.

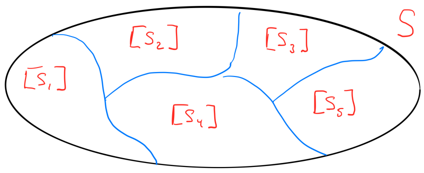

Figure 1.1: The set S is split into disjoint equivalence classes

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Conjugation

Returning to groups, here is one way to define an equivalence relation on a group.

Definition 1.3. *For a group G, two elements $g_1, g_2 \in G$ are called **conjugate** if*

$$\exists\, g \in G \mid g_2 = g g_1 g^{-1} \tag{1.1}$$

*This is an equivalence relation with equivalence classes called **conjugacy classes** and explicitly given by*

$$[g_1] = \{g_2 \in G \mid g_2 = g g_1 g^{-1} \text{ for some } g \in G\} \tag{1.2}$$

Exercise 1.2. *Show that conjugation defines an equivalence relation.*

Conjugation and conjugacy classes will play an important role when we discuss representations of finite groups in the next chapter.

Subgroups and cosets

After introducing the basic algebraic structure one of the next standard steps is to define the sub-structure of groups, that is, subgroups.

Definition 1.4. *(Subgroups) A* ***subgroup*** *H of G is a subset $H \subset G$ which is itself a group under the group multiplication induced from G. Every group G has the trivial subgroups $\{e\}$ and G. All other subgroups of G are called* ***proper subgroups****.*

Exercise 1.3. *Show that a subset $H \subset G$ of a group G is a subgroup if and only if it has the following properties: (i) It is closed under multiplication, that is $hh' \in H \; \forall \; h, h' \in H$ (ii) $e \in H$ (iii) $\forall h \in H \;\; \Rightarrow \;\; h^{-1} \in H$.*

For a group G and a subgroup $H \subset G$, we can define another equivalence relation

$$g_1 \sim g_2 \quad \Leftrightarrow \quad g_1^{-1} g_2 \in H \tag{1.3}$$

Definition 1.5. *(Cosets) The equivalence classes with regard to the equivalence relation (1.3), given by $gH = \{gh \mid h \in H\}$, are called a* ***(left) cosets*** *of H in G.*

For a finite group G, every coset gH has the same number of elements (the same number of elements as H) since multiplication by a fixed $g \in G$ is an invertible map. If we denote the number of elements of a finite group G, also called the **order of the group**, by $|G|$ we can draw two interesting conclusions from this.

$$(i) \quad |G| = k|H| \text{ where } k \in \mathbb{N} \tag{1.4}$$

$$(ii) \quad \text{If } |G| \text{ is prime, then } G \text{ has no proper subgroups.} \tag{1.5}$$

If we consider a finite group G and an element $g \in G$, then not all power $e = g^0, g^1, g^2, \ldots$ can be different. This means we must have $g^k = g^l$ and, hence $g^{k-l} = e$, for some positive integers k, l with $k > l$. The smallest integer p for which $g^p = e$ is called the **order** of the group element g.

Normal subgroups

In much the same way that we have introduced left cosets above we can also introduce **right cosets**, simply by changing the order of multiplication in the definition. There is a particularly important class of subgroups for which the left and right cosets are identical.

Definition 1.6. *(Normal subgroups) A subgroup $H \subset G$ is called* ***normal*** *iff $gH = Hg$ for all $g \in G$, that is, if left and right cosets are the same.*

If $H \subset G$ is a normal subgroup, then the set of all cosets of H in G, the **quotient space**

$$G/H = \{gH \mid g \in G\} \tag{1.6}$$

forms a group under the multiplication

$$(g_1 H) \cdot (g_2 H) = (g_1 g_2) H \; . \tag{1.7}$$

Exercise 1.4. *Show that the quotient space G/H, where $H \subset G$ is a normal subgroup, forms a group under the multiplication (1.7). What is the neutral element of this group?*

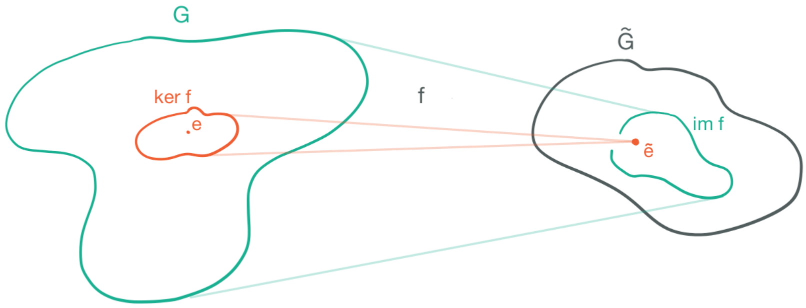


Figure 1.2: Illustration of the isomorphism theorem.

Group homomorphisms

The next basic step is to define the maps consistent with the algebraic structure, the group homomorphisms.

Definition 1.7. *(Homomorphism) A map $f : G \to \tilde{G}$ is called a **(group) homomorphism** iff*

$$f(g_1 g_2) = f(g_1) f(g_2) \quad \forall\, g_1, g_2 \in G \tag{1.8}$$

*The set of all group homomorphisms $G \to \tilde{G}$ is also denoted by $\mathrm{Hom}(G, \tilde{G})$. The **kernel** and **image** of a group homomorphism are defined as*

$$\mathrm{Ker}(f) = \{g \in G \mid f(g) = \tilde{e}\} \subset G \tag{1.9}$$

$$\mathrm{Im}(f) = \{f(g) \mid g \in G\} \subset \tilde{G} \tag{1.10}$$

*where $\tilde{e}$ is the identity element of $\tilde{G}$. If the homomorphism $f : G \to \tilde{G}$ is bijective it is called a (group) **isomorphism**. If such an isomorphism exists between G and $\tilde{G}$ we say the groups are **isomorphic** and we write $G \cong \tilde{G}$.*

Remark 1.2. *Group homomorphisms $G \to G$ are also called **(group) endomorphisms** of G and the set of these is denoted $\mathrm{End}(G) = \mathrm{Hom}(G, G)$. Invertible endomorphisms are called **(group) automorphisms** of G and the set $\mathrm{Aut}(G)$ of all automorphisms $G \to G$ forms a group (with multiplication the composition of maps) called the **automorphisms group** of G.*

Remark 1.3. *Group homomorphisms have the following basic properties.*

- $f(e) = \tilde{e}$ *and* $f(g^{-1}) = f(g)^{-1}$
- *Ker(f) is a normal subgroup of G. This implies the quotient $G/$ Ker(f) is a group.*
- *Im(f) is a subgroup of $\tilde{G}$*
- *f is one-to-one (**injective**) $\Leftrightarrow$ Ker(f) $= \{e\}$*
- *f is onto (**surjective**) $\Leftrightarrow$ Im(f) $= \tilde{G}$*

Exercise 1.5. *Prove the properties of group homomorphisms in the previous remark.*

The above remarks show that the obstruction to a homormophism $f : G \to \tilde{G}$ being injective is a non-trivial kernel $\text{Ker}(f) \neq \{e\}$. This obstruction can be removed by passing to the quotient $G/\text{Ker}(f)$. On the other hand, we can make the map surjective by replacing the co-domain $\tilde{G}$ with $\text{Im}(f)$ (which is a sub-group of $\tilde{G}$). These observation suggest the following theorem.

Theorem 1.6. *(Isomorphism theorem) If $f : G \to \tilde{G}$ is a group homomorphism, then*

$$G/\text{Ker}(f) \cong \text{Im}(f) \tag{1.11}$$

Exercise 1.7. *Prove the isomorphism theorem. To do this, first show that the homomorphism $f : G \to \tilde{G}$ induces an obvious homomorphism $\hat{f} : G/\text{Ker}(f) \to \text{Im}(f)$. Then show that $\hat{f}$ is bijective.*

1.2 Representations

We are now ready to define the central mathematical structure of these lectures, the **(linear) representations** of a group and explore their basic properties. An important part of the discussion will be to define **irreducible representations** which form the basic building blocks and can be used to construct new representations. We end by discussing several ways in which given representations can be used to define new ones. These are closely aligned with the standard methods of defining new vector spaces from old ones: the dual vector space and tensor vector spaces. If your linear algebra on these topics is hazy you might want to remind yourself.

Definition of representations

Recall that our goal was to define a mathematical structure which facilitates the action of groups on vector spaces. The first step is to observe that the set $\text{GL}(V)$ $(= \text{Aut}(V))$ of invertible linear maps $V \to V$ on an n-dimensional vector space V over a field $\mathbb{F} = \mathbb{R}, \mathbb{C}$ forms a group, called the **general linear group** of V. The group multiplication is composition of maps, the unit element is the unit map and the inverse is the map inverse. For $V = \mathbb{F}^n$ the general linear group consists of invertible $n \times n$ matrices with entries in $\mathbb{F}$, the group multiplication is matrix multiplication, the group identity is the unit matrix and the group inverse is the matrix inverse.

By definition the general linear group $\text{GL}(V)$ acts (linearly) on the vector space V. An obvious way for an arbitrary group G to act on a vector space is to assign to each group element in G a linear transformation in $\text{GL}(V)$. Of course this assignment should "preserve" the structure of the group G, that is, it should be a group homomorphism $G \to \text{GL}(V)$. This motivates the following definition.

Definition 1.8. *(Representation) A **representation** R of a group G is a group homomorphism $R : G \to \text{GL}(V)$ where V is a vector space over $\mathbb{F}$. The **dimension of the representation** R is defined by $\dim(R) = \dim_{\mathbb{F}}(V)$.*

Remark 1.4. *(i) Often we consider the vector spaces $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$. In this case, $\text{GL}(V)$ is the group of invertible $n \times n$ matrices (with real or complex entries) and the $R(g)$ are matrices from this group.*

(ii) The homomorphism property is crucial and it means that $R(g_1 g_2) = R(g_1)R(g_2)$, so that the representation matrices multiply in the "same way" (have the same multiplication table)

as the group elements. This property is hinted at by the terminology "representation".

(iii) The fact that R is a group homomorphism implies that $R(g^{-1}) = R(g)^{-1}$ and $R(e) = \mathrm{id}_V$.

Equivalence of representations

As ever, when a new class of mathematical objects has been defined we need to think about under which circumstances we want to consider two such objects as "equal" or equivalent. If the representation matrices of two representations are related by a common basis transformation we should consider the two representations equivalent - after all, the choice of basis on a vector space is arbitrary. Generalising this idea slightly leads to this definition.

Definition 1.9. *(Equivalent representations) Two representations $R_1 : G \to \mathrm{GL}(V_1)$ and $R_2 : G \to \mathrm{GL}(V_2)$ are called **equivalent** iff these exists an isomorphism $\phi : V_1 \to V_2$ such that*

$$R_1(g) = \phi^{-1} \circ R_2(g) \circ \phi \qquad\qquad \forall\, g \in G\,. \tag{1.12}$$

*In this case, we write $R_1 \cong R_2$. Otherwise, the representations are called **inequivalent** and we write $R_1 \ncong R_2$.*

Remark 1.5. *(i) Note that the linear map ϕ in the above definition is the same for all $g \in G$. Its existence (or otherwise) is, therefore, a non-trivial matter.*

(ii) Since $\phi : V_1 \to V_2$ is an isomorphism (so $\dim_{\mathbb{F}}(V_1) = \dim_{\mathbb{F}}(V_2)$ in particular) two representations R_1 and R_2 can only be equivalent if they have the same dimension, $\dim(R_1) = \dim(R_2)$.

Basis properties of representations

Often vector spaces V have additional structure, the most prominent one being a s scalar product, $\langle\cdot,\cdot\rangle$. A vector space with a scalar product is also called an **inner product vector space**. On such an inner product space we can consider the invertible linear maps $f \in \mathrm{GL}(V)$ which leave the scalar product invariant, that is, the maps $f \in \mathrm{GL}(V)$ which satisfy $\langle f(v), f(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$. Such linear maps are called **unitary** and the set of all unitary linear maps (relative to the given scalar product), denoted $\mathrm{U}(V)$, forms a subgroup of $\mathrm{GL}(V)$. If $V = \mathbb{R}^n$ with the dot product, then the unitary maps are the orthogonal $n \times n$ matrices, denoted $\mathrm{O}(n) = \mathrm{U}(\mathbb{R}^n)$. For $V = \mathbb{C}^n$ with the standard hermitian scalar product the unitary maps are precisely the unitary $n \times n$ matrices $\mathrm{U}(n) = \mathrm{U}(\mathbb{C}^n)$.

Exercise 1.8. *Let V be an inner product vector space. Show that the set $\mathrm{U}(V)$ of unitary maps forms a subgroup of $\mathrm{GL}(V)$.*

For an inner product vector space V we can consider special representations where all representation maps are contained in the subgroup $\mathrm{U}(V) \subset \mathrm{GL}(V)$.

Definition 1.10. *(Unitary representations) A representation $R : G \to \mathrm{GL}(V)$ on an inner product vector space V is called **unitary** if $R(g)$ is a **unitary** linear map for all $g \in G$ (or, equivalently, iff $\mathrm{Im}(R) \subset \mathrm{U}(V)$).*

The following is just a somewhat redundant but often-used piece of terminology.

Definition 1.11. *(Faithful representation) A representation $R : G \to \mathrm{GL}(V)$ is called **faithful** iff it is one-to-one, or, equivalently, iff $\mathrm{Ker}(R) = \{e\}$.*

For a faithful representation, the representation maps $R(g)$ capture the full group structure - hence the name. For non-faithful representations two or more group elements are mapped to the same representation map and information about the group structure "is lost".

Reducible and irreducible representations

Now we come to the crucial notion of a **reducible/irreducible representation** which is the key to understanding how representations can be decomposed into basic building blocks.

Definition 1.12. *(Reducible and irreducible representations) A representation $R : G \to \mathrm{GL}(V)$ is called* ***reducible*** *iff a non-trivial subspace $U \subset V$ ($U \neq \{0\}, V$) exists such that $R(g)U \subset U \;\forall\, g \in G$ (that is, the subspace U is invariant under the group action). Otherwise, R is called* ***irreducible*** *or an* ***irrep****.*

To get to an intuitive idea of what this definition means consider a reducible representation R and choose a basis of V by starting with a basis of U and completing this to a basis of V. Relative to such a basis, $R(g)$ is described by a matrix $[R(g)]$ of the form

$$[R(g)] = \left(\begin{array}{cc} A(g) & B(g) \\ 0 & C(g) \end{array}\right) \qquad \begin{array}{l} \text{basis of } U \\ \text{other basis vectors not in } U \end{array} \tag{1.13}$$

The zero entry arises due to the closure of U under R which implies that a vector $(u, 0)^T$ in U must be transformed into a vector of the same form. Unfortunately, for vectors $(0, w)^T$ there is no such requirement and this is why we cannot conclude that the matrices $B(g)$ vanish. The presence of the matrix $B(g)$ is an obstruction to decomposing a reducible representation and cases where all $B(g)$ vanish are called fully reducible. The proper definition of this term is as follows.

Definition 1.13. *(Fully reducible representations) Let $R : G \to \mathrm{GL}(V)$ be a representation. If there exists a direct sum decomposition $V = U_1 \oplus \cdots \oplus U_k$ such that $R(g)(U_i) \subset U_i$ for all $g \in G$ and all $i = 1, \ldots, k$ and the representations $R_i : G \to \mathrm{GL}(U_i)$ defined by $R_i(g) = R(g)|_{U_i}$ are irreducible then R is called* ***fully reducible****. In this case, we write $R = R_1 \oplus \cdots \oplus R_k$.*

To see what this means it is again useful to think of a basis adapted to the direct sum decomposition $V = U_1 \oplus \cdots \oplus U_k$. Relative to such a basis the representation matrices are of the block-diagonal form

$$[R(g)] = \left(\begin{array}{ccc} [R_1(g)] & & 0 \\ & \ddots & \\ 0 & & [R_k(g)] \end{array}\right) \qquad \begin{array}{c} U_1 \\ \vdots \\ U_k \end{array} \tag{1.14}$$

Exercise 1.9. *Suppose that all representation matrices of a representation R are of the block-diagonal form (1.14). Show that every block $[R_i(g)]$ defines a representation.*

In other words, a fully reducible representation R can be written as a direct sum, $R = R_1 \oplus \cdots \oplus R_k$ of irreducible representations R_i. In this case we have the dimension formula

$$\dim(R) = \sum_{i=1}^{k} \dim(R_i)\ . \tag{1.15}$$

Note that an irreducible representation is trivially fully reducible.

Given this discussion, we would like to know when a reducible representation is fully reducible. Unfortunately, this is not always the case but it turns out that important classes of representations are fully reducible. The first of these are unitary representations.

Proposition 1.2. *Unitary representations are fully reducible.*

Proof. We consider a representation $R : G \to \mathrm{U}(V)$ on a vector space with inner product $\langle \cdot, \cdot \rangle$ (recall, the notion of unitary representation is only defined if we have an inner product). If R is irreducible then it is fully reducible and there is nothing to show.

Let us assume R is reducible so that $R(g)U \subset U$ for a subspace $U \subset V$. Define the perpendicular space $W = U^\perp = \{w \in V \mid \langle w, u \rangle = 0 \; \forall \; u \in U\}$ and recall from standard linear algebra that $V = U \oplus W$.

Consider $w \in W$. It follows that $\langle R(g)w, u \rangle = \langle w, R(g)^{-1}u \rangle = 0$ since $R(g)^{-1}u \in U$ and, therefore, $R(g)w \in U^\perp = W$. This means the perpendicular space W is also invariant under R and, hence, the matrices $B(g)$ in Eq. (1.13) vanish.

If both $R|_U$ and $R|_W$ are irreducible we are finished. Otherwise, we apply the above argument to $R|_U$ and $R|_W$ and continue until we are left with irreducible subspaces only. □

Many of the representations relevant in physics are unitary, so this covers significant ground. Finite groups are another important class with fully reducible representations.

Corollary 1.1. *Any representation of a finite group is unitary (relative to a suitable scalar product) and, hence, fully reducible.*

Proof. On V introduce an arbitrary scalar product $\langle \cdot, \cdot \rangle_0$ (that this is possible is a standard result of linear algebra). Define a new scalar product by

$$\langle v, w \rangle = \sum_{\tilde{g} \in G} \langle R(\tilde{g})v, R(\tilde{g})w \rangle_0 . \tag{1.16}$$

Note that the sum is well-defined since the group G is finite. This new scalar product is invariant under all $R(g)$, that is,

$$\langle R(g)v, R(g)w \rangle = \langle v, w \rangle \qquad \forall \; v, w \in V \tag{1.17}$$

since the left action of the representation on both vectors merely permutes the terms in the sum in (1.16). It follows that the representation R is unitary relative to the scalar product $\langle \cdot, \cdot \rangle$ and, therefore, from the previous proposition, that it is fully reducible. □

Exercise 1.10. *Prove the invariance* (1.17) *of the scalar product* (1.16) *under the representation R.*

Schur's Lemma

Schur's Lemma constrains linear maps which commute with all representation maps of an irreducible representation. You have probably used a special version of Schur's Lemma in Quantum Mechanics ("If the Hamilton operator commutes with angular momentum, then angular momentum eigenstates $|jm\rangle$ have the same energy for all m.") The general statement is as follows.

Lemma 1.1. *(Schur's Lemma) Let $R : G \to GL(V)$ and $\tilde{R} : G \to GL(W)$ be two irreducible representations over complex vector spaces V and W. Further, let $\phi : V \to W$ be a linear map satisfying $\phi \circ R(g) = \tilde{R}(g) \circ \phi$ for all $g \in G$. Then, we have the following statements.*

(i) Either ϕ is an isomorphism or $\phi = 0$.
(ii) If $V = W$ and $R = \tilde{R}$, then $\phi = \lambda\,\mathrm{id}_V$ where $\lambda \in \mathbb{C}$.

Proof. (i) We begin by showing that $\mathrm{Ker}(\phi)$ is invariant under R. Consider a $v \in \mathrm{Ker}(\phi)$, so that $\phi(v) = 0$. Then $\phi(R(g)v) = \tilde{R}(g)(\phi(v)) = 0$ and it follows that $R(g)v \in \mathrm{Ker}(\phi)$. This means that $\mathrm{Ker}(\phi)$ is invariant under R. In the same way, we can show that $\mathrm{Im}(\phi)$ is invariant under $\tilde{R}$. But R and $\tilde{R}$ are irreducible so Ker $(\phi) = \{0\}$ or V and Im $(\phi) = \{0\}$ or W. Suppose that $\mathrm{Ker}(\phi) = V$. In this case, $\phi = 0$ and the second possibility in (i) is realised. On the other hand, if $\mathrm{Ker}(\phi) = \{0\}$ then ϕ is injective and its image cannot be trivial. Hence, $\mathrm{Im}(\phi) = W$ and ϕ is surjective.

(ii) Define the eigenspace $\mathrm{Eig}_\phi(\lambda) = \{v \in V \mid \phi(v) = \lambda v\}$ of ϕ with eigenvalue λ. This eigenspace is non-trivial for at least one λ, since $\det(\phi - \lambda \mathrm{id}_V) = 0$ has a solution in $\mathbb{C}$. For this value of λ consider an eigenvector $v \in \mathrm{Eig}_\phi(\lambda)$. The short calculation $\phi(R(g)v) = R(g)\phi(v) = \lambda R(g)v$ shows that $R(g)v \in \mathrm{Eig}_\phi(\lambda)$ and, hence, that $\mathrm{Eig}_\phi(\lambda)$ is invariant under R. Since R is irreducible and $\mathrm{Eig}_\phi(\lambda) \neq \{0\}$ it follows that $\mathrm{Eig}_\phi(\lambda) = V$. This means that $\phi = \lambda\,\mathrm{id}_V$. □

It is worth restating Schur's Lemma slightly. Suppose that the first option in (i) is realised and ϕ is an isomorphism. Then $R(g) = \phi^{-1} \circ \tilde{R}(g) \circ \phi$ which shows that R and $\tilde{R}$ are equivalent in this case. Let us fix an isomorphism $\phi_0 : V \to W$ which realises this equivalence, so that $R(g) = \phi_0^{-1} \circ \tilde{R}(g) \circ \phi_0$. We would then like to determine the set of all maps ϕ which "intertwine" R and $\tilde{R}$, that is, $\phi \circ R(g) = \tilde{R}(g) \circ \phi$. If we define $\psi = \phi_0^{-1} \circ \phi : V \to V$ an easy calculation shows that $\psi \circ R(g) = R(g) \circ \psi$, so we can apply part (ii) of Schur's Lemma and conclude that $\psi = \lambda\,\mathrm{id}_V$ or, equivalently, $\phi = \lambda\,\phi_0$. On the other hand, if R and $\tilde{R}$ are inequivalent then the second possiblity in (i) must be realised and $\phi = 0$. We can summarise this discussion by saying that, under the assumptions of Schur's Lemma, the map ϕ is given by

$$\phi = \begin{cases} 0 & \text{if} \quad R \ncong \tilde{R} \\ \lambda\,\phi_0,\ \lambda \in \mathbb{C} & \text{if} \quad R \cong \tilde{R} \end{cases} \tag{1.18}$$

where $\phi_0 : V \to W$ is any isomorphism which realises the equivalence, so $R(g) = \phi_0^{-1} \circ \tilde{R}(g) \circ \phi_0$.

Part (ii) of Schur's Lemma is the more widely known statement. In short, it says

$$[\phi, R(g)] = 0 \quad \forall g \in G \qquad \Rightarrow \qquad \phi = \lambda\,\mathrm{id}_V \tag{1.19}$$

provided R is an irreducible representation over a complex vector space. (Here, $[f, g] := f \circ g - g \circ f$ is the commutator.) "A linear map which commutes with all representation maps of a complex irrep must be a multiple of the identity map."

Schur's Lemma leads to an interesting statement about the representations of Abelian groups.

Corollary 1.2. *All complex irreps of Abelian groups are one-dimensional.*

Proof. Let G be Abelian and $R : G \to \mathrm{Gl}(V)$ be a complex irrep of G. Commutativity of the group multiplication implies $[R(g), R(g')] = 0\ \forall\ g, g' \in G$. If we set $\phi = R(g)$ we can apply part (ii) of Schur's Lemma and conclude that $R(g) = \lambda(g)\,\mathrm{id}_V$. However, a representation of this form is only irreducible if $\dim(V) = 1$. □

New representations from old ones

We now quickly review a few standard constructions of representations. Most of these are based on standard constructions for vector spaces, such as direct sum, dual vector spaces and tensor vector spaces.

Trivial representation

This doesn't quite fit the bill but needs to be mentioned. The trivial representation $R : G \to \mathrm{GL}(V)$ is defined by

$$R(g) = \mathrm{id}_V \quad \forall g \in G \; . \tag{1.20}$$

This representation is of course not faithful and if $\dim(V) > 1$ then it is reducible. The one-dimensional trivial representation, $\dim(V) = 1$, is irreducible so it appears in the list of irreducible representations for every group.

Direct sum representation

This is basically the opposite of decomposing a fully reducible representation into its irreducible pieces. Suppose, we have representations $R_i : G \to \mathrm{GL}(U_i)$, for $i = 1, \ldots, k$. The direct sum vector space $V = U_1 \oplus \cdots \oplus U_k$ consists of all vectors of the form $v = u_1 + \cdots + u_k$, where $u_i \in U_i$. We can define a representation $R : G \to \mathrm{GL}(V)$ on the direct sum by

$$R(g)(u_1 + \cdots + u_k) := R_1(g)(u_1) + \cdots + R_k(g)(u_k). \tag{1.21}$$

Relative to a basis of V which is adapted to the direct sum (choosing a basis for each U_i and combining all those vectors to a basis of V) the representation matrices for $R(g)$ have a block-diagonal form, as in Eq. (1.14).

Exercise 1.11. *Use the definition (1.21) to show that the direct sum representation is indeed a representation, that is, show that $R(g_1 g_2) = R(g_1)R(g_2)$.*

Reminder: Dual and tensor vector spaces

This is a "rough-and-ready" summary of the relevant facts without paying much attention to mathematical niceties. If you haven't seen some of this before you might want to consult a linear algebra book.

- Start with a vector space V over the field $\mathbb{F}$ and a basis (e^i), so that any $v \in V$ is expressed as $v = v_i e^i$.

- The **dual vector space** $V^* = \mathrm{Hom}(V, \mathbb{F})$ is the space of linear forms on V - that is, linear maps $V \to \mathbb{F}$. For a basis $\{e_i\}$ of V^* we can write $\lambda = \lambda^i e_i$ for $\lambda \in V^*$. This basis can be chosen to be the **dual basis** to the basis (e^i) of V which is defined by the relations $e_i(e^j) = \delta_i^j$. For this choice, the action of a functional $\lambda \in V^*$ on a vector $v \in V$ can be written as $\lambda(v) = \lambda^i v_i$. Note that a vector space V and its dual V^* have the same dimension. (If you have encountered vectors with upper and lower indices before and have wondered what they are, then here is the answer. They correspond to vectors in a vector space and in its dual.)

- For the **tensor vector space** we consider two vector spaces V and W with bases (e^i) and (ϵ^a), respectively. The tensor product $V \otimes W$ is spanned by the vectors $v \otimes w$ (a bi-linear operation), where $v \in V$ and $w \in W$, and it has a basis $(e^i \otimes \epsilon^a)$. Hence, every tensor $t \in V \otimes W$ can be written as $t = t_{ia} e^i \otimes \epsilon^a$ and we have

$$\dim_{\mathbb{F}}(V \otimes W) = \dim_{\mathbb{F}}(V) \dim_{\mathbb{F}}(W) \ . \tag{1.22}$$

We can form multiple tensor products with more than two factors and, in particular, we can consider the tensor product $V^{\otimes p} \otimes (V^*)^{\otimes q}$ which consists of p factors of V and q factors of its dual V^*. Tensors in this space are also called (p,q)-tensors and they can be written as

$$T = T_{i_1,\dots,i_p}{}^{j_1,\dots j_q} \ e^{i_1} \otimes \cdots \otimes e^{i_p} \otimes e_{j_1} \otimes \cdots \otimes e_{j_q} \tag{1.23}$$

- The space $\mathrm{Hom}(V,W)$ of linear maps $V \to W$ is a vector space (which is isomorphic to the vector space of matrices with size $\dim(W) \times \dim(V)$) with dimension

$$\dim_{\mathbb{F}}(\mathrm{Hom}(V,W)) = \dim_{\mathbb{F}}(V) \dim_{\mathbb{F}}(W) \ . \tag{1.24}$$

The relation between a linear map $f \in \mathrm{Hom}(V,W)$ and its representing matrix M is via the equation

$$f(e^i) = M_a{}^i \epsilon^a \quad \leftrightarrow \quad M_a{}^j \epsilon^a \otimes e_j \tag{1.25}$$

which shows that $\mathrm{Hom}(V,W) \cong W \otimes V^*$.

The original representation

Let us start by working out the action of a representation $R : G \to \mathrm{GL}(V)$ on the components v_i of a vector $v = v_i e^i$. Since

$$R(g)(v) = v_i R(g)(e^i) = [R(g)]_i{}^j v_j e^i \tag{1.26}$$

the components transform as $v_i \mapsto [R(g)]_i{}^j v_j$, where $[R(g)]_i{}^j$ is the matrix representing $R(g)$.

Dual representation

For a representation $R : G \to \mathrm{GL}(V)$ on a vector space V there is a natural associated representation $R' : G \to \mathrm{GL}(V^*)$, called the **dual representation**, defined by

$$(R'(g)\lambda)(R(g)v) = \lambda(v) \quad \forall v \in V,\ \forall \lambda \in V^* \qquad \Leftrightarrow \qquad R'(g) = R(g)^{-1^T} \tag{1.27}$$

For $\lambda = \lambda^i e_i$ we have $R'(g)(\lambda) = [R'(g)]^i{}_j \lambda^j e_i$ so the components transform as $\lambda^i \mapsto [R'(g)]^i{}_j \lambda^j$ with $[R'(g)]^i{}_j = [R(g)^{-1}]_j{}^i$. Of course the representation R and its dual representation R' have the same dimension, $\dim(R) = \dim(R')$.

The **complex conjugate representation** R^* of a matrix representation $R : G \to \mathrm{GL}(\mathbb{C}^n)$ is defined as $R^*(g) = R(g)^*$. For unitary matrix representations $R : G \to \mathrm{GL}(\mathbb{C}^n)$ we have $R(g)^{-1^T} = R(g)^{\dagger^T} = R(g)^*$ so that the dual representation R' is, in fact, the same as the complex conjugate representation. A matrix representation R is called **real** if there exists a basis such that $R^*(g) = R(g)$ for all $g \in G$, that is, if all representation matrices are real matrices. If R and R^* are inequivalent representations then R is called **complex**. There is also an intermediate case, when R and R^* are equivalent but no basis can be found such that $R^*(g) = R(g)$ for all $g \in G$. In this case, R is called **pseudo-real**.

Exercise 1.12. *Show that the dual R' of a representation R, as defined in Eq. (1.27), is indeed a representation. Convince yourself that the maps $g \mapsto R(g)^{-1}$ and $g \mapsto R(g)^T$ do not define representations.*

Tensor product representation

For representations $R_V : G \to GL(V)$ and $R_W : G \to GL(W)$ on vector spaces V and W there is a natural representation $R_V \otimes R_W : G \to \mathrm{GL}(V \otimes W)$, called the **tensor representation** of R_V and R_W, defined by

$$(R_V \otimes R_W)(g)(v \otimes w) \equiv R_V(g)(v) \otimes R_W(g)(w) \; . \tag{1.28}$$

From Eq. (1.22), we know the dimension of the tensor representation is

$$\dim(R_V \otimes R_W) = \dim(R_V)\dim(R_W) \; . \tag{1.29}$$

It is worth working out what this means relative to a basis, so we use the above definition (plus linearity) on a tensor $t = t_{ia} e^i \otimes \epsilon^a$ and work out

$$(R_V \otimes R_W)(g)(t) = t_{ia} R_V(g)(e^i) \otimes R_W(g)(\epsilon^a) = [R_V(g)]_j{}^i [R_W(g)]_b{}^a t_{ia} e^j \otimes \epsilon^b \; . \tag{1.30}$$

Hence, the tensor components transform as $t_{jb} \mapsto [R_V(g)]_j{}^i [R_W(g)]_b{}^a t_{ia}$. Note that this is in complete analogy with the transformation of a vector, except that we require a transformation matrix per index.

It also shows that the entries of the matrix associated to $R_V \otimes R_W(g)$ contains products of the matrix elements from $R_V(g)$ and $R_W(g)$. To write this down more explicitly we introduce the **Kronecker product** of an $n \times n$ matrix A and an $m \times m$ matrix B by

$$A \times B := \begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & \cdots & \cdots \\ \vdots & \ddots & \vdots \end{pmatrix} \; . \tag{1.31}$$

In other words, the Konecker product $A \times B$ is an $(nm) \times (nm)$ matrix obtained by replacing every entry A_{ij} of A with $A_{ij}B$, that is, with the entire matrix B times that entry. Useful properties of the Kronecker product are

$$A \times (B + C) = A \times B + A \times C \; , \qquad (A \times B)(C \times D) = (AC) \times (BD) \; . \tag{1.32}$$

If we now order the basis $(e_i \otimes \epsilon_a)$ of the tensor space $V \otimes W$ as $(e^1 \otimes \epsilon^1, e^1 \otimes \epsilon^2, \ldots, e^1 \otimes \epsilon^{\dim(W)}, e^2 \otimes \epsilon^1, \ldots)$ then the matrix for $R_V \otimes R_W(g)$ is given by the Kronecker product

$$[R_V \otimes R_W(g)] = [R_V(g)] \times [R_W(g)] \; . \tag{1.33}$$

Exercise 1.13. *Proof the relations (1.32) for the Kronecker product. Use these to show, from Eq. (1.33), that the tensor representation is indeed a representation.*

In fact, the above construction can be generalised somewhat to two groups G_1 and G_2 with representations $R_1 : G_1 \to \mathrm{GL}(V_1)$ and $R_2 : G_2 \to \mathrm{GL}(V_2)$. In this situation, the tensor product can be used to define a representation $R : G_1 \times G_2 \to \mathrm{GL}(V_1 \otimes V_2)$ of the direct product group $G_1 \times G_2$ by $R((g_1, g_2))(v_1 \otimes v_2) = R_1(g_1)(v_1) \otimes R_2(g_2)(v_2)$. The above discussion of tensor representations immediately generalises, with the group arguments suitably adjusted. In particular, the tensor transformation in components is given by Eq. (1.30).

Induced representation on linear maps

For R_V and R_W as given above there is an induced representation $R_{\mathrm{Hom}(V,W)} : G \to \mathrm{GL}(\mathrm{Hom}(V,W))$ This follows from the fact that $\mathrm{Hom}(V,W) \cong W \otimes V^*$ which also shows that

$$R_{\mathrm{Hom}(V,W)}(g)(f) = R_W(g) \circ f \circ R_V(g)^{-1} \; . \tag{1.34}$$

The dimension of this representation is, of course,

$$\dim(R_{\mathrm{Hom}(V,W)}) = \dim(R_V)\dim(R_W) \; . \tag{1.35}$$

Clebsch-Gordan decomposition

Start with two irreducible representations $R : G \to \mathrm{GL}(V)$ and $\tilde{R} : G \to \mathrm{GL}(W)$. It is by no means clear that the tensor representation $R \otimes \tilde{R}$ is also irreducible and, as we will see later, it is usually a reducible representation. If it is fully reducible, then we can write

$$R \otimes \tilde{R} = \bigoplus_s R_s \; , \tag{1.36}$$

where the R_s are irreducible representations. An equation of the form (1.36) is known as a **Clebsch-Gordan decomposition** - it encodes which irreducible representation are contained in a tensor product. In this case, we have the useful relation

$$\dim(R \otimes \tilde{R}) = \dim(R)\dim(\tilde{R}) = \sum_s \dim(R_s) \tag{1.37}$$

between the various dimensions. If the dimensions of the irreducible representations are known this relation can be helpful in narrowing down which irreps R_s can appear in the Clebsch-Gordan decomposition.

Induced representation on subgroups and branching

If $H \subset G$ is a subgroup, we can restrict the representation $R^{(G)} : G \to \mathrm{GL}(V)$ of G to a representation $R^{(H)} : H \to \mathrm{GL}(V)$ of H, simply by using the same representation maps, so $R^{(H)}(h) = R^{(G)}(h) \quad \forall\, h \in H$. In this way, every representation of a group induces a representation of a subgroup.

Suppose that $R^{(G)}$ is irreducible. It is by no means clear that the induced representation $R^{(H)}$ remains irreducible. There might well be a subspace $U \subset V$ which is invariant under $R^{(H)}$ but not under the generally larger set of representation maps from $R^{(G)}$. So, in general, $R^{(H)}$ does not have to be irreducible and we have a decomposition, also referred to as **branching** or **branching rule**, given by

$$R^{(G)} \to R^{(H)} = \bigoplus_s R_s^{(H)} \; , \tag{1.38}$$

where $R_s^{(H)}$ are irreps of H. Such branching rules play a role in physics whenever a symmetry H of a physical system is embedded into a larger symmetry G.

1.3 Examples

We are now ready to discuss the two simplest examples of Abelian groups and determine their irreducible representations. The key statement which facilitates this is Corollary 1.2 which asserts that the complex irreps of such Abelian groups are one-dimensional.

$\underline{G = \mathbb{Z}_n}$

This is the simplest finite Abelian group, also called the **cyclic group**. It is defined as $\mathbb{Z}_n = \{0, 1, ..., n-1\}$ with the group operation $g_1 \cdot g_2 = (g_1 + g_2) \bmod n$. This is an Abelian group, thus all irreps over complex vector spaces are one-dimensional, from Corollary 1.2.

Consider such an irrep

$$R : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^* = \mathbb{C}\backslash\{0\} .$$

The representation matrices are (invertible) 1×1 matrices, that is, non-zero complex numbers. We have

$$R(1)^n = R(1^n) = R(0) = 1$$

(remembering that the group "multiplication" is, in fact, addition) which implies that $R(1)$ is an n^{th} root of unity, so $R(1) = \exp\left[-2\pi i q/n\right]$ where $q = 0, ..., n-1$. For each choice of q we get an irrep which we call R_q. Since $R_q(g) = R_q(1)^g$ it follows that

$$R_q(g) = \exp\left[-2\pi i q\; g/n\right] , \qquad q = 0, \ldots, n-1 .$$

This provides explicitly the n irreps of $\mathbb{Z}_n$ and, in a physics context, the integer q which labels the representations is also called the **charge**. Note we have found the number of irreducible representations is finite and also that the $q = 0$ representation is the trivial one.

$\underline{G = U(1)}$

The group $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in the complex plane with group operation the usual multiplication of complex numbers. It is an Abelian group so its irreducible representations (over complex vector spaces) are one-dimensional.

Start with such a representation $R : U(1) \to \mathbb{C}^*$ and consider a unit-length complex number of the form $z = e^{2\pi i p/q}$, where $p, q \in \mathbb{Z}$. For such z is follows that

$$z^q = 1 \implies R(z)^q = 1 \implies |R(z)| = 1 .$$

For complex numbers $z = \exp(i\phi)$ with an arbitrary real phase ϕ the same statement holds if we demand continuity of R (by using the fact that every real number can be obtained as a limit of rational ones). It follows that the image of R is contained in $U(1) \subset \mathbb{C}^*$ and that we can write

$$R(e^{i\phi}) = e^{ir(\phi)} ,$$

for some function r. The representation property of R translates into the functional equation $r(\phi_1 + \phi_2) = r(\phi_1) + r(\phi_2)$ which shows that r is a linear function and can, hence, be written as $r(\phi) = -q\phi$ for some real constant q. This means we have representations R_q, given by $R_q(\exp(i\phi)) = \exp(-iq\phi)$. We also demand that the representation is continuous at $z = 1$ which means it should approach the same value for $\phi = \epsilon$ and $\phi = 2\pi - \epsilon$, as $\epsilon \to 0$. This means

$$1 = R_q(1) = R_q\left(\lim_{\phi\to 2\pi} e^{i\phi}\right) = \lim_{\phi\to 2\pi} R_q(e^{i\phi}) = \lim_{\phi\to 2\pi} e^{-iq\phi} = e^{-2\pi i q} ,$$

so that $q \in \mathbb{Z}$. In summary, the (continuous) irreducible complex representations of $U(1)$ are indexed by an integer q, in physics also called the **charge**, and are explicitly given by

$$R_q(e^{i\phi}) = e^{-iq\phi} , \qquad q \in \mathbb{Z} . \tag{1.39}$$

Hence, the number of complex irreducible $U(1)$ representations is countably infinite.

Exercise 1.14. *Show that $R : U(1) \to \mathrm{GL}(\mathbb{R}^2)$ defined by*

$$R_p(e^{i\phi}) = \begin{pmatrix} \cos(p\phi) & -\sin(p\phi) \\ \sin(p\phi) & \cos(p\phi) \end{pmatrix}$$

for $p \in \mathbb{Z}$ is an irreducible representation. Why does this not contradict Corollary 1.2?

Application 1.1: *($U(1)$ and $\mathbb{Z}_n$ in field theory model building)*

Calling the integer q which labels $U(1)$ and $\mathbb{Z}_n$ representations "charge" is not a misnomer. In a physics context it actually represents the charge as, for example, in " charge of the electron" (modulo the fact that, for historical reasons, to do with the late discovery of quarks, physical charges are taken to be elements of $\mathbb{Z}/3$ rather than $\mathbb{Z}$).

To see how this works more practically, consider fields Ψ_i, where $i = 1, \ldots, k$, in a field theory which, at each point in space(-time), take values in $\mathbb{C}$. From a mathematical point of view we should think of Ψ_i as typical elements in a vector space $V_i = \mathbb{C}$. In a physics context this is often not made very explicit - the underlying (vector) space is implied by the use of a typical element. Now suppose that on V_i we have a representation of $\mathbb{Z}_n$ or $U(1)$ with charge q_i, so that we have the group actions

$$\mathbb{Z}_n : \quad \Psi_i \mapsto \exp(-2\pi i q_i g/n)\Psi_i \ , \qquad U(1) : \quad \Psi_i \mapsto \exp(-iq_i\phi)\Psi_i \tag{1.40}$$

on the fields Ψ_i. In this case we say that the field Ψ_i (or the particle associated with it in a quantum field theory) has charge q_i. A common problem is to construct terms in a Lagrangian which are invariant under a symmetry. Consider the term $\Psi_1 \cdots \Psi_k$. Its transformation is

$$\mathbb{Z}_n : \quad \Psi_1 \cdots \Psi_k \mapsto \exp\left(-\frac{2\pi i g}{n}\sum_i q_i\right)\Psi_1 \cdots \Psi_k \tag{1.41}$$

$$U(1) : \quad \Psi_1 \cdots \Psi_k \mapsto \exp\left(-i\phi\sum_i q_i\right)\Psi_1 \cdots \Psi_k \tag{1.42}$$

Another way to express these transformation laws is to say that the term $\Psi_1 \cdots \Psi_k$ has charge $\sum_i q_i \bmod n$ in the $\mathbb{Z}_n$ case and charge $\sum_i q_i$ in the $U(1)$ case. This means $\Psi_1 \cdots \Psi_k$ is invariant (that is, corresponds to the trivial representation) iff

$$\mathbb{Z}_n : \quad \sum_i q_i = 0 \bmod n \ , \qquad U(1) : \quad \sum_i q_i = 0 \ , \tag{1.43}$$

that is, iff the charge sums up to zero.

The groups $\mathbb{Z}_n$ and $U(1)$ are prototypes for the large classes of groups we will study in the remainder of these lectures: finite groups and Lie groups.

Application 1.2: *(Restricting scalar field potentials)*

To be a little more concrete, suppose we have a field theory with a complex-valued scalar field $\phi : \mathbb{R}^4 \to \mathbb{C}$ and we would like to use $U(1)$ or $\mathbb{Z}_n$ symmetries to constrain its scalar potential $V(\phi, \bar{\phi})$. There are a number of constraints on this scalar potential which we would like to impose even before invoking symmetries: it is real (so the Lagrangian is), it is polynomial (not strictly necessary but the usual starting point in QFTs), we are not considering linear terms (can be shifted away) and the highest order is four (this is to restrict to renormalisable terms in four dimensions). On this basis the most general expression is

$$V(\phi,\bar{\phi}) = m^2|\phi|^2 + \lambda|\phi|^4 + \left(\frac{1}{2}\mu^2\phi^2 + \frac{1}{3!}\kappa\phi^3 + \frac{1}{4!}\xi\phi^4 + \text{c.c.}\right) + \left(\nu|\phi|^2\phi + \frac{1}{2}\rho|\phi|^2\phi^2 + \text{c.c.}\right) .$$

How can this potential be restricted further by imposing symmetries? First, note that neither $U(1)$ nor $\mathbb{Z}_n$ symmetries can forbid the first two terms: since ϕ and $\bar{\phi}$ have opposite charges the combination $|\phi|^2$ is always invariant. What can be done about the other terms?

First consider imposing a $\mathbb{Z}_2$ symmetry which acts as $\phi \mapsto -\phi$. This forces $\kappa = \nu = 0$ but allows all other terms. Next suppose ϕ transforms under the charge 1 representation of $\mathbb{Z}_3$, so $\phi \mapsto \alpha\phi$, where $\alpha = \exp(2\pi i/3)$. Since $\alpha^3 = 1$ the κ-term is now allowed but we are forced to set $\mu = \xi = \nu = \rho = 0$. Similarly, if ϕ carries charge 1 under a $\mathbb{Z}_4$ symmetry then $\mu = \kappa = \nu = \rho = 0$ but the ξ-term is allowed. If ϕ carries charges 1 under a $\mathbb{Z}_n$ symmetry with $n > 4$ or under $U(1)$ then $\mu = \kappa = \xi = \nu = \rho = 0$.

We will tackle finite groups in the next chapter and then move to Lie algebras but in either case we will, of course, not demand that the group is Abelian. This makes it quite hard to copy the "brute-force" approach we have taken for the above examples which essentially amounts to writing down an Ansatz for the representation matrices and impose that they obey the correct group multiplication laws. For Abelian groups this works since the matrices for irreducible representations are 1×1 matrices so just numbers. These are easy to deal with and, in addition, there is no ambiguity due to basis choice as in Def. 1.9.

For non-Abelian groups these advantages fall away. Imagine writing down an Ansatz for a, say, 2×2 representation matrix for every group element and then trying to fix, by "calculation", the matrix entries so that the right multiplication rules are obeyed. This becomes intractable quickly and, due to the ambiguity in basis choice, this will not even completely fix the matrices. This is a good example for how the common physicists' method of "putting-your-head-through-the-wall-by-calculation" can be quite useless. Some problems need a deeper, more structured approach which recognises the nature of the problem. We will now introduce such an approach for finite groups.

Chapter 2

Finite groups

In this chapter we consider finite groups G and their representations.

2.1 Characters

An important feature to remember about representations is the notion of equivalence in Def. 1.9. We are not interested in finding all irreducible representations, but we would rather like to find all inequivalent, irreducible ones. Since equivalence corresponds to basis transformations we should, therefore, build basis-independence into our method. You already know at least two basis-invariants of linear maps: the trace and the determinant. The trace is much simpler - it is linear in the entries of a matrix - and it turns out, luckily, that it is the correct object to tackle our problem. This motivates the definition of **characters**.

Definition and basis properties

Definition 2.1. *Let $R : G \to GL(V)$ be a representation of G. The* ***character*** *$\chi_R : G \to \mathbb{C}$ of the representation R is defined by*

$$\chi_R(g) = \mathrm{tr}(R(g)). \tag{2.1}$$

Remark 2.1. *(i) Note that χ_R is constant on conjugacy classes (Def. 1.3), making it an example of a* ***class function****. This follows from the cyclicity of the trace and the representation property.*

$$\chi_R(hgh^{-1}) = \mathrm{tr}(R(hgh^{-1})) = \mathrm{tr}(R(h)R(g)R(h)^{-1}) = \mathrm{tr}(R(g)) = \chi_R(g)\ .$$

More generally, a class function is a function $\alpha : G \to \mathbb{C}$ with $\alpha(hgh^{-1}) = \alpha(g)$ for all $h, g \in G$.

(ii) The character value of the group identity is the dimension of the representation.

$$\chi_R(e) = \mathrm{tr}(R(e)) = \mathrm{tr}(\mathrm{id}_V) = \dim(R)$$

We require a few rules for how to calculate with characters.

Proposition 2.1. *For representations R, $\tilde{R}$ of G the characters satisfy*

(1) $\chi_{R\oplus\tilde{R}}(g) = \chi_R(g) + \chi_{\tilde{R}}(g)$
(2) $\chi_{R\otimes\tilde{R}}(g) = \chi_R(g)\chi_{\tilde{R}}(g)$
(3) $\chi_{R'}(g) = \chi_R(g)^*$

Proof. (1) For a direct sum of representations, the representation matrices have the form

$$[R \oplus \tilde{R}(g)] = \begin{pmatrix} [R(g)] & 0 \\ 0 & [\tilde{R}(g)] \end{pmatrix}$$

and (i) follows from $\mathrm{tr}([R \oplus \tilde{R}(g)]) = \mathrm{tr}([R(g))] + \mathrm{tr}([\tilde{R}(g)])$.

(2) For a suitable choice of basis, the representation matrices for the tensor product are given by the Kronecker product and the claim follows from $\mathrm{tr}(A \times B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$ where $\times$ is the Kronecker product (see Section 1.2).

(3) We can diagonalise $R(g) \to [R(g)] = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and, since $g^k = e$ for the order $k \in \mathbb{N}$ of g, it follows that the λ_i must be k^{th} roots of unity. Then, $[R'(g)] = [R(g)^{-1^T}] = \mathrm{diag}(\lambda_1^*, ..., \lambda_n^*) = [R(g)]^*$ (given that $z^{-1} = z^*$ for a root of unity z). It follows that $\mathrm{tr}(R(g))^* = \mathrm{tr}(R'(g))$ which is the desired statement. □

Singlets in representations

If a representation $R : G \to GL(V)$ is reducible it can be decomposed into its irreducible pieces. The simplest irreducible representations are (one-dimensional) trivial representations and the associated one-dimensional vector subspaces are also referred to as **singlets**. The space of singlets in V is given by

$$V^G = \{v \in V \mid R(g)v = v \; \forall \, g \in G\} \qquad \text{(singlets)} \; . \tag{2.2}$$

A closely related linear map $p : V \to V$ is defined by

$$p = \frac{1}{|G|} \sum_{g \in G} R(g) : V \to V \; , \tag{2.3}$$

and this map turns out to be a projector onto the singlet space.

Proposition 2.2. *The linear map $p : V \to V$ in Eq. (2.3) is a projector onto V^G. In other words, we have (a) $\mathrm{Im}(p) = V^G$ and (b) $p \circ p = p$.*

Proof. (a) We show this set equality by mutual inclusion.
$\mathrm{Im}(p) \subset V^G$: Let $v \in \mathrm{Im}(p)$ so that $v = p(w)$ for some $w \in V$. Then

$$v = p(w) = \frac{1}{|G|} \sum_{g \in G} R(g)w \; .$$

We would like to show that v is in the singlet space and to do this we prove its invariance under G.

$$R(\tilde{g})v = \frac{1}{|G|} \sum_{g \in G} R(\tilde{g}g)w = \frac{1}{|G|} \sum_{g \in G} R(g)w = v$$

It follows that $v \in V^G$ and, hence, $\mathrm{Im}(p) \subset V^G$.
$V^G \subset \mathrm{Im}(p)$: To show the reverse conclusion we start with a $v \in V^G$, so that $R(g)v = v$ for all $g \in G$. it follows that

$$p(v) = \frac{1}{|G|} \sum_{g \in G} R(g)v = v \; , \tag{2.4}$$

so v is its own image under p and, hence, $v \in \mathrm{Im}(p)$. This proves the inclusion $V^G \subset \mathrm{Im}(p)$. It follows that $\mathrm{Im}(p) = V^G$ which completes (a).

(b) Set $w = p(v)$ for $v \in V$ and note that $w \in \mathrm{Im}(p) = V^G$ and, hence, $p(w) = w$ from Eq. (2.4). This means $p \circ p(v) = p(w) = w = p(v)$ which proves (b). $\square$

Note there is a simple formula for the dimension of the singlet space V^G ("the number of singlets")

$$\dim(V^G) = \mathrm{tr}(p) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(R(g)) = \frac{1}{|G|} \sum_{g \in G} \chi_R(g) \tag{2.5}$$

in terms of the character.

Orthonormality of characters

The above results provide some insight into the trivial representation, but of course we would like information about all irreps. The trick is to start with two complex irreps $R_V : G \to \mathrm{GL}(V)$ and $R_W : G \to \mathrm{GL}(W)$ and apply the above results for singlets to the representation induced on $\mathrm{Hom}(V, W)$. From Eq. (1.34) we have

$$\begin{aligned} \mathrm{Hom}(V, W)^G &= \left\{ \phi \in \mathrm{Hom}(V, W) \mid R_{\mathrm{Hom}(V,W)}(g)\phi = R_W(g) \circ \phi \circ R_V(g)^{-1} = \phi \right\} \\ &= \{ \phi \in \mathrm{Hom}(V, W) \mid R_W(g) \circ \phi = \phi \circ R_V(g) \} \end{aligned} \tag{2.6}$$

From Schur's Lemma in the form (1.18) the dimension of this space can be computed as

$$\dim_{\mathbb{F}} \left(\mathrm{Hom}\,(V, W)^G \right) = \left\{ \begin{array}{lll} 1 & \text{for} & R_V \cong R_W \\ 0 & \text{for} & R_V \ncong R_W \end{array} \right.$$

while Eq. (2.5) together with $\chi_{\mathrm{Hom(V,W)}}(g) = \chi_{V^* \otimes W}(g) = \chi_V^*(g)\chi_W(g)$ gives

$$\dim_{\mathbb{F}} \left(\mathrm{Hom}(V, W)^G \right) = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V^*(g)\chi_W(g) \, .$$

Equating these two results for $\dim_{\mathbb{F}} \left(\mathrm{Hom}(V, W)^G \right)$ leads to a key result for the characters of irreps.

Theorem 2.1. *Relative to the* ***inner product of characters*** *defined by*

$$(\chi, \tilde{\chi}) = \frac{1}{|G|} \sum_{g \in G} \chi(g)^* \tilde{\chi}(g) \tag{2.7}$$

the characters χ_i of the irreducible representations R_i form an orthonormal system, that is, $(\chi_i, \chi_j) = \delta_{ij}$.

In practice, we can think of a character χ as a vector $(\chi(e), \chi(g_1), \chi(g_2), \ldots)$ in $\mathbb{C}^{|G|}$ which contains the values of χ on all group elements. In this case, the inner product (2.7) simply becomes the standard hermitian scalar product on $\mathbb{C}^{|G|}$. But the character is a class function, so it is better to think of a character as a vector in $\mathbb{C}^{n_C}$, where n_C is the number of conjugacy classes of the group G ("one value per class"). Since there cannot be more than n_C orthonormal vectors in $\mathbb{C}^{n_C}$ we immediately have the following

Corollary 2.1. *The number of irreps is less than or equal to the number of conjugacy classes of G.*

In particular, this means that the number of irreps is finite, something which was not clear before. We will prove later that the number of irreps is, in fact, equal to the number of conjugacy classes.

Representation content from characters

The ortho-normality of the irrep characters is a powerful tool which can be used to analyse the irrep content of an arbitrary representation.

Corollary 2.2. *Let $R_1, ..., R_k$ be the irreps of G, $\chi_1, ..., \chi_k$ the associated characters, and R an arbitrary representation of G. Since R is fully reducible we can write $R \cong R_1^{\oplus m_1} \oplus \cdots \oplus R_k^{\oplus m_k}$, so that R contains m_i copies of the irrep R_i. Then we have the following statements.*

(i) R is completely determined by its character χ_R.
(ii) The multiplicity m_i of R_i in R is given by $m_i = (\chi_R, \chi_i)$.
(iii) R is an irrep iff $(\chi_R, \chi_R) = 1$.

Proof. (i) This follows from part (ii). The representation R is fully determined if we know its irrep content and, from (ii) this can be computed from the character χ_R.

(ii) To show (ii) we start with $R = R_1^{\oplus m_1} \oplus \cdots \oplus R_k^{\oplus m_k}$ which implies $\chi_R = \sum_{j=1}^{k} m_j \chi_j$. It follows that

$$(\chi_R, \chi_i) = \sum_{j=1}^{k} m_j \underbrace{(\chi_j, \chi_i)}_{=\delta_{ij}} = m_i$$

and this is the required formula.

(iii) R is an irrep iff one of the $m_i = 1$ and all the others are zero. Since

$$(\chi_R, \chi_R) = \sum_{i,j=1}^{k} m_i m_j (\chi_i, \chi_j) = \sum_{i=1}^{k} m_i^2$$

this is equivalent to $(\chi_R, \chi_R) = 1$. □

2.2 The regular representation

As we have seen, characters help us to work out the irrep content of given representations but this assumes we already know the irreps and their characters. How do we find them in the first place? It turns out the crucial mathematical structure required is the group algebra.

Group algebra and regular representation

Definition 2.2. *The **group algebra** A_G of G is the set of formal linear combinations*

$$v = \sum_{g \in G} v(g) g \ , \qquad v(g) \in \mathbb{C}$$

where all $g \in G$ are considered to be linearly independent. Hence, $\dim(A_G) = |G|$. *(An algebra is a vector space with a bi-linear multiplication. Vector addition and scalar multiplications in A_G are defined component-wise in the obvious way. The multiplication is induced by the group multiplication, using its bi-linearity.).*

The point of this definition is that we can construct a canonical representation of G on its own group algebra A_G.

Definition 2.3. *The **regular representation** $R_{\rm reg} : G \to {\rm GL}(A_G)$ is the representation of the group on its own algebra, defined by*

$$R_{\rm reg}(g)v = gv \; . \tag{2.8}$$

Note that the right-hand-side of the definition (2.8) is simply a multiplication in the algebra A_G. The dimension of the regular representation

$$\dim(R_{\rm reg}) = \dim(A_G) = |G| \tag{2.9}$$

equals the order, $|G|$, of the group. It turns out that the regular representation contains every irrep.

Exercise 2.2. *Write down the group algebra $A_{\mathbb{Z}_2}$ and work out its multiplication.*

Representation content of regular representation

Theorem 2.3. *Let $R_1, \ldots, R_k$ be the irreps of G. Then we have $R_{\rm reg} = R_1^{\oplus \dim(R_1)} \oplus \cdots \oplus R_k^{\oplus \dim(R_k)}$. This means the irrep R_i appears dim(R_i) times and, in particular, every irrep is contained in $R_{\rm reg}$. The dimensions of the irreps satisfy*

$$\sum_{i=1}^{k} (\dim(R_i))^2 = |G| \tag{2.10}$$

Proof. Clearly, the dimension formula (2.10) is implied by the first part of the theorem, given that $\dim(R_{\rm reg}) = |G|$. To prove the first part we start by computing the character of the regular representation.

$$R_{\rm reg}(g)\tilde{g} = g\tilde{g} \left\{ \begin{array}{lll} = \tilde{g} & \text{for} & g = e \\ \neq \tilde{g} & \text{for} & g \neq e \end{array} \right. \quad \Rightarrow \quad \chi_{\rm reg}(g) = \left\{ \begin{array}{lll} \dim(R_{\rm reg}) = |G| & \text{for} & g = e \\ 0 & \text{for} & g \neq e \end{array} \right.$$

To understand why ${\rm tr}(R_{\rm reg}(g)) = 0$ for $g \neq e$, write $G = \{g_i \,|, i = 1, \ldots, |G|\}$, so label the group elements by an index. The (g_i) form a basis of A_G on which we can evaluate $R_{\rm reg}(g)$ by writing

$$R_{\rm reg}(g)(g_j) = \sum_i [R(g)]^i{}_j g_i = g g_j \; .$$

Provided $g \neq e$, the product gg_j is a group element different from g_j so all diagonal entries of the matrix $[R(g)]$ must be zero and, hence, its trace vanishes.

Now we write the regular representation as $R_{\rm reg} = R_1^{\oplus m_1} \oplus \cdots \oplus R_k^{\oplus m_k}$ and compute the multiplicities m_i of the irreps using the previous theorem.

$$m_i = (\chi_{\rm reg}, \chi_i) = \frac{1}{|G|} \sum_{g \in G} \chi^*_{\rm reg}(g)\, \chi_i(g) = \frac{1}{|G|} \chi^*_{\rm reg}(e)\, \chi_i(e) = \frac{1}{|G|} |G| \dim(R_i) = \dim(R_i)$$

□

The main implications of this theorem can be summarised as follows.

- The regular representation contains all irreps of G, so we have to "reduce" $R_{\rm reg}$ to find the irreps explicitly.
- Each irrep is contained in $R_{\rm reg}$ with a multiplicity that equals its dimension.
- Eq. (2.10) provides a strong dimensional constraint on irreps.

Exercise 2.4. *What are the possible dimensions of the irreps of a group of order 6?*

2.3 Examples

It is time to practice the use of characters with a few examples, starting by revisiting the Abelian case.

$\underline{G = \mathbb{Z}_3}$

This is the group $\mathbb{Z}_3 = \{0, 1, 2\}$ with group multiplication defined by addition modulo 3. Its order is $|\mathbb{Z}_3| = 3$. We already know that is has three irreps R_q, where $q = 0, 1, 2$, which are all one-dimensional, and are given by

$$R_q(g) = \alpha^{qg} \ , \qquad \alpha \equiv e^{2\pi i/3} \ .$$

Since the group is Abelian, every element forms its own conjugacy class, so we have the three classes $C_0 = \{0\}$, $C_1 = \{1\}$ and $C_2 = \{2\}$. For one-dimensional representations, the characters are identical to the representations, $\chi_q = R_q$. Putting this together, the **character table** for $\mathbb{Z}_3$ is given by

	C_0	C_1	C_2
χ_0	1	1	1
χ_1	1	α	α^2
χ_2	1	α^2	$\alpha^4 = \alpha$

Orthonormality, $(\chi_i, \chi_j) = \delta_{ij}$, of these characters follows from the relation $1 + \alpha + \alpha^2 = 0$, for example

$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)^* \chi_2(g) = \frac{1}{3} \left(1 \times 1 + \alpha^{-1} \times \alpha^2 + \alpha^{-2} \times \alpha\right) = \frac{1}{3} \left(1 + \alpha + \alpha^2\right) = 0$$

To see how these characters can be used to extract the irrep content of a representation consider the three-dimensional representation $R : \mathbb{Z}_3 \to \mathrm{GL}(\mathbb{C}^3)$ defined by

$$R(g) = A^q \ , \qquad A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ . \tag{2.11}$$

This is indeed a representation since $A^3 = \mathbb{1}_3$. Since $\mathrm{tr}(A) = \mathrm{tr}(A^2) = 0$ its character is given by $\chi_R = (3, 0, 0)$. Taking the scalar product of this character with the ones for the irreps from the above table gives

$$(\chi_R, \chi_i) = \frac{1}{3} \sum_{g=0}^{2} \chi_R(g)^* \chi_i(g) = 1$$

Recall that these are precisely the multiplicities of the irreps in R, so we have $R = R_0 \oplus R_1 \oplus R_2$, that is, R is the regular representation (it contains every irrep with multiplicity given by its dimension).

The group algebra $A_{\mathbb{Z}_3}$ is spanned by the three group elements, which we denote by bold face numbers $(\mathbf{0}, \mathbf{1}, \mathbf{2})$ in order to avoid confusion with actual numbers. Then the group algebra is

$$A_{\mathbb{Z}_3} = \{a\, \mathbf{0} + b\, \mathbf{1} + c\, \mathbf{2} \mid a, b, c \in \mathbb{C}\}$$

and it is equipped with a multiplication based on

$$\begin{array}{rclcl} \mathbf{1}\cdot\mathbf{0} & = & \mathbf{1} & = & 0\mathbf{0}+1\mathbf{1}+0\mathbf{2} \\ \mathbf{1}\cdot\mathbf{1} & = & \mathbf{2} & = & 0\mathbf{0}+0\mathbf{1}+1\mathbf{2} \\ \mathbf{1}\cdot\mathbf{2} & = & \mathbf{0} & = & 1\mathbf{0}+0\mathbf{1}+0\mathbf{2} \end{array}$$

The coefficients in each row on the right-hand side form the columns of the matrix representing $R_{\rm reg}(1)$ relative to the basis $(\mathbf{0},\mathbf{1},\mathbf{2})$ and, hence, we have $[R_{\rm reg}(1)] = A$. This confirms that the three-dimensional representation (2.11) is indeed the regular representation.

Exercise 2.5. *Find the characters of the $\mathbb{Z}_n$ irreps and show that they are ortho-normal. Show that $R : \mathbb{Z}_n \to {\rm GL}(\mathbb{C}^n)$ defined by $R(g)(e_i) = e_{(i+g) \bmod n}$, where e_i are the standard unit vectors, defines a representation. Find the irrep content of R.*

Quaternion group

The **quaternion group** is a group of order eight with elements $\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$ and group multiplication defined by the relations

$$i^2 = j^2 = k^2 = -1\ , \qquad ij = -ji = k \quad \text{and cyclic} \tag{2.12}$$

A bit of playing around with these relations reveals there are five conjugacy classes

$$C_1 = \{1\}\ , \quad C_2 = \{-1\}\ , \quad C_i = \{\pm i\}\ , \quad C_j = \{\pm j\}\ , \quad C_k = \{\pm k\}\ . \tag{2.13}$$

For example, the equivalence of i and $-i$ follows from $jij^{-1} = -jij = -jk = -i$.

Given that we have 5 conjugacy classes we know from Cor. 2.1 that there are at most five irreps. The other information about these irreps is the dimensional constraint (2.10) which reads

$$\sum_{i=1}^{k\leq 5} (\dim R_i)^2 = |\mathbb{H}| = 8\ .$$

It seems one possible solution is to have just two irreps, each with dimension two, but this does not leave any room for the trivial representation which always exist. Given this, the only possibility is to have four one-dimensional and one two-dimensional irrep.

It is not hard to work out the one-dimensional representations. To construct the two-dimensional representation we can use the **Pauli matrices** σ_i, where $i = 1, 2, 3$, defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{2.14}$$

These matrices square to one, $\sigma_i^2 = \mathbb{1}_2$, and two of them multiply to (i times) the third, for example $\sigma_1\sigma_2 = i\sigma_3$. These properties can be summarised by the relation

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1}_2 + i\epsilon_{ijk}\sigma_k\ . \tag{2.15}$$

For the computation of characters it is useful to note that the Pauli matrices have vanishing trace, ${\rm tr}(\sigma_i) = 0$. The rules (2.15) are very similar to the multiplication relations (2.12) for the quaternion group and the differences in factors can be fixed by using $i\sigma_i$ instead of σ_i. In summary, this gives the representations and characters as in the following tables.

g	± 1	$\pm i$	$\pm j$	$\pm k$
R_1	1	1	1	1
R_i	1	1	-1	-1
R_j	1	-1	1	-1
R_k	1	-1	-1	1
R_2	$\pm \mathbb{1}_2$	$\pm i\sigma_3$	$\pm i\sigma_2$	$\pm i\sigma_1$

	C_1	C_2	C_i	C_j	C_k
# elements	1	1	2	2	2
χ_1	1	1	1	1	1
χ_i	1	1	1	-1	-1
χ_j	1	1	-1	1	-1
χ_k	1	1	-1	-1	1
χ_2	2	-2	0	0	0

These characters are indeed orthonormal with regard to the product $(\cdot,\cdot)$. (But note that to verify this the number of elements in each conjugacy class has to be taken into account.)

Consider the four-dimensional representation R_4 of $\mathbb{H}$ defined by $R_4(\pm 1) = \pm \mathbb{1}_4$ and

$$R_4(\pm i) = \pm \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \quad R_4(\pm j) = \pm \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \quad R_4(\pm k) = \pm \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Exercise 2.6. *Verify that R_4 is indeed a representation of $\mathbb{H}$ by checking the multiplication relations* (2.12).

What is the representation content of R_4? Taking the trace of the above matrices we find $\chi_{R_4} = (4, -4, 0, 0, 0)$, so that

$$(\chi_{R_4}, \chi_1) = (\chi_{R_4}, \chi_i) = (\chi_{R_4}, \chi_j) = (\chi_{R_4}, \chi_k) = 0 \ , \quad (\chi_{R_4}, \chi_{R_2}) = 2 \qquad \Rightarrow \qquad R_4 = R_2 \oplus R_2 \ .$$

It is quite remarkable how easily this result is obtained. We now know that there must be a common basis transformation which brings the above 4×4 matrices into a block-diagonal form with each of the two 2×2 blocks forming a representation R_2. But finding this transformation explicitly looks like significantly more effort than the above character calculation.

This is also a good opportunity to practice computing a Clebsch-Gordan decomposition. What is the irrep content of $R = R_2 \otimes R_2$? Since, $\chi_R(g) = \chi_2(g)^2$ (see Prop. 2.1) we have $\chi_R = (4, 4, 0, 0, 0)$, so that

$$(\chi_R, \chi_{1,i,j,k}) = 1 \ , \quad (\chi_R, \chi_2) = 0 \qquad \Rightarrow \qquad R_2 \otimes R_2 = R_1 \oplus R_i \oplus R_j \oplus R_k \ .$$

Application 2.1: *(Yukawa model building with finite groups)*
As a model building application, consider a Yukawa term of the form

$$\lambda_{ij} H \bar{\psi}^i_L \psi^j_R \tag{2.16}$$

where H is a complex scalar field and ψ^i_L, ψ^i_R are left- and right-handed fermions and $i = 1, 2, 3$ is a family index. (Details of the structure of these fields will be discussed later, in the section on the Lorentz group, but this will not be important for the present discussion.) The fields might be the quarks, $(\psi^i_L, \psi^i_R) = (u^i_L, u^i_R)$ or (d^i_L, d^i_R) or the leptons $(\psi^i_L, \psi^i_R) = (e^i_L, e^i_R)$. The Yukawa couplings times the vacuum expectation value of the field H (the Higgs field) lead to a mass matrix, $M_{ij} = \lambda_{ij}\langle H\rangle$, which determines the masses (and mixings) of the particles.

One avenue of model building, aiming at an explanation of fermion masses and mixings, is to constrain the couplings (2.16) by imposing a discrete symmetry under which H, ψ^i_L and ψ^i_R

transform. Without any ambition of producing a realistic model, we would like to illustrate how this works in principle, using the quaternion group $\mathbb{H}$.

From its character table it is easy to extract the Clebsch-Gordan decompositions

$$\begin{array}{rclcrcl} R_p \otimes R_p &=& R_1 & \qquad & R_i \otimes R_j &=& R_k \text{ and cyclic} \\ R_p \otimes R_2 &=& R_2 & & R_2 \otimes R_2 &=& R_1 \oplus R_i \oplus R_j \oplus R_k \end{array}$$

where $p = i, j, k$, and this information is the basis for building a model. To be specific, assume the following assignment of representations

$$H \sim R_i\ , \quad \bar{\psi}^3_L \sim R_j\ , \quad \psi^3_R \sim R_k\ , \quad \bar{\chi}_L := (\bar{\psi}^1_L, \bar{\psi}^2_L) \sim R_2\ , \quad \chi_R := (\psi^1_R, \psi^2_R) \sim R_2\ .$$

In other words, we arrange the first two families, for both left- and right-handed fields, into the two-dimensional representations and all other fields into one-dimensional ones. (There are of course many other ways to do this.) With this assignment we should only keep the terms in Eq. (2.16) which are $\mathbb{H}$-invariant. Since $H\bar{\psi}^3_L\chi_R \sim R_i \otimes R_j \otimes R_2 = R_2$ we learn immediately that $\lambda_{31} = \lambda_{32} = 0$. For similar reasons, $\lambda_{13} = \lambda_{23} = 0$. So right away, the Yukawa matrix is restricted to

$$\lambda = \begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$$

where the $\star$ indicates a potentially non-zero entry. In fact, the λ_{33} entry is allowed since $R_i \otimes R_j \otimes R_k = R_1$. What about the 2×2 block? Since $H \sim R_i$ only the representation $R_i \in R_2 \times R_2$ is allowed. In fact, the one-dimensional subspace which corresponds to $R_i \in R_2 \otimes R_2$ is given by $\bar{\chi}^T_L \sigma_2 \chi_R$, as an explicit R_2 transformation shows. (Transform with the non-trivial R_2 representation matrices $\pm i\sigma_i$ as $\bar{\chi}_L \mapsto \pm i\sigma_i \bar{\chi}_L$ and $\chi_R \mapsto \pm i\sigma_i \chi_R$.)

Hence, the final form of the Yukawa matrix is

$$\lambda = \begin{pmatrix} 0 & \lambda_1 & 0 \\ -\lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\ ,$$

where λ_1, λ_3 are arbitrary.

Exercise 2.7. *Try other charge assignments of the quaternion group to constrain the Yukawa terms (2.16). Also, try other finite groups to do the same.*

2.4 More on the regular representation

There are a few loose ends to tie up. We still need to show that the number of irreps equals the number of conjugacy classes. It would also be useful to have more information about the structure of the regular representation and how to extract the irreps from it. We begin with a lemma on some useful properties of class functions.

The number of irreps

Lemma 2.1. *Let $\alpha : G \to \mathbb{C}$ be a class function, $R : G \to GL(V)$ a complex representation and the map $\phi_{\alpha,R} : V \to V$ is defined by $\phi_{\alpha,R} = \sum_{g\in G} \alpha(g)R(g) : V \to V$. Then*

(i) $[\phi_{\alpha,R}, R(h)] = 0$ *for all* $h \in G$

(ii) If R is an irrep then $\phi_{\alpha,R} = \lambda \mathbb{1}_{dim(V)}$

Proof. (i) The vanishing of this commutator can be shown by an explicit calculations. For $v \in V$ we have

$$\begin{aligned}\phi_{\alpha,R}(R(h)v) &= \sum_{g\in G}\alpha(g)R(gh)v = \sum_{g\in G}\alpha(hgh^{-1})R(hgh^{-1}h)v \\ &= R(h)\left(\sum_{g\in G}\alpha(g)R(g)v\right) = R(h)\left(\phi_{\alpha,R}v\right)\end{aligned}$$

(ii) This follows from part (i) and Schur's Lemma. $\square$

Theorem 2.8. *The number of irreps of G equals the number of equivalence classes.*

Proof. Let R_i be the irreps of G with associated characters χ_i, where $i = 1, \ldots, k$. Recall that we can think of a character as a vector in $\mathbb{C}^{n_C}$, where n_C is the number of conjugacy classes of G. If we can show that the irrep characters span the entire space $\mathbb{C}^{n_C}$, then it follows that $k = n_C$, which is the desired statement. We can do this by proving that any class function $\alpha : G \to \mathbb{C}$ which is perpendicular to all irrep characters must be zero. So we have to show that $(\alpha, \chi_i) = 0$ for $i = 1, \ldots, k$ implies $\alpha = 0$.
To this end, we define the maps $\phi_{\alpha,i} = \sum_{g\in G}\alpha(g)^* R_i(g)$. Part (ii) of the previous Lemma implies that $\phi_{\alpha,i} = \lambda\mathbb{1}$, for some $\lambda \in \mathbb{C}$. In fact, λ can be computed from

$$\lambda = \frac{1}{\dim(R_i)}\mathrm{tr}(\phi_{\alpha,i}) = \frac{1}{\dim(R_i)}\sum_{g\in G}\alpha^*(g)\chi_i(g) = \frac{|G|}{\dim(R_i)}(\alpha,\chi_i) = 0$$

This means that $\phi_{\alpha,i} = 0$ for all $i = 1, \ldots, k$. But since every representation R can be written as a sum of irreps it follows that $\phi_{\alpha,R} = 0$. Consider the regular representation:

$$0 = \phi_{\alpha,\mathrm{reg}} = \sum_{g\in G}\alpha^*(g)g\ .$$

Since the group elements g are linearly independent as a basis of A_G it follows that $\alpha(g) = 0$ for all $g \in G$. Hence, $\alpha = 0$, which is what we needed to show. $\square$

Structure of regular representation

Recall the structure of the regular representation is

$$R_{\mathrm{reg}} = R_1^{\oplus \dim R_1} \oplus \cdots \oplus R_k^{\oplus \dim R_k}\ ,$$

so the multiplicity of every irrep in R_{reg} equals its dimension. The group algebra A_G must have a corresponding decomposition into vector subspaces $V_{i,j}$ with

$$A_G = V_1 \oplus \cdots \oplus V_k\ , \qquad V_i = V_{i,1} \oplus \cdots \oplus V_{i,\dim(R_i)}\ , \qquad \dim(V_{i,j}) = \dim(R_i)\ . \tag{2.17}$$

Each subspace V_i contains the $\dim(R_i)$ copies of the representation R_i and, therefore, decomposes further into $\dim(R_i)$ subspaces $V_{i,j}$ each hosting one representation R_i. If we can find the subspaces $V_{i,j}$ then we can extract the irreps from the regular representation. The following theorem is a partial solution to this problem in that it provides projectors for the subspaces V_i.

Theorem 2.9. *The maps $P_i : A_G \to A_G$ defined by*

$$P_i = \frac{\dim(R_i)}{|G|} \sum_{g\in G} \chi_i(g)^* g : A_G \to A_G \tag{2.18}$$

are projectors onto the subspace $V_i \subset A_G$ in Eq. (2.17).

Proof. Denote the irreps and their character by R_i and χ_i, where $i = 1, \ldots, k$, and define the maps

$$\psi_{ij} \equiv \frac{\dim(R_j)}{|G|} \sum_{g\in G} \chi_i^*(g) R_j(g) \ .$$

Since the characters are class functions we know from Lemma 2.1 that $\psi_{ij} = \lambda \mathbb{1}$. As before, we can compute λ by taking the trace.

$$\lambda = \frac{1}{\dim(R_j)} \mathrm{tr}(\psi) = \frac{1}{|G|} \sum_{g\in G} \chi_i^*(g)\, \mathrm{tr}(R_j(g)) = (\chi_i, \chi_j) = \delta_{ij} \ .$$

This means that

$$P_i|_{V_{j,k}} = \psi_{ij} = \begin{cases} \mathbb{1} & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

Hence, P_i is a projector onto V_i. $\square$

2.5 More examples

Quaternion group (continued)

If we insert characters of for the quaternion group $\mathbb{H}$ in Section 2.3 into Eq. (2.18) we can obtain projectors onto the various subspaces of the group algebra. For example, a projector onto the subspace which corresponds to the one-dimensional representation R_i is given by

$$P_{R_i} = \frac{1}{8}\left(1 + (-1) + i + (-i) - j - (-j) - k - (-k)\right) =: g$$

It is easy to check that P_{R_i} does indeed project onto a one-dimensional subspace, namely $\mathrm{Span}(g) \subset A_{\mathbb{H}}$. The representation "matrices" (in this case just numbers since we are considering a one-dimensional irrep) can be obtained by acting with the elements of $\mathbb{H}$ on g and reading off the pre-factors.

$$\pm 1 g = g \ , \quad \pm i g = g \ , \quad \pm j g = -g \ , \quad \pm k g = -g \ .$$

The signs on the right-hand sides of these equations do indeed correctly reproduce the values of $R_i(g)$, as comparison with the table in Section 2.3 shows. In this same way, we obtain the projector associated to the two-dimensional representation R_2, which is given by $P_{R_2} = \frac{1}{2}(1 - (-1))$. It projects onto a four-dimensional subspace of $A_{\mathbb{H}}$ since a two-dimensional irrep in contained twice in the group algebra.

The permutation group S_n

There is much to say about permutation groups and their representations. However, we need to keep things brief so we will only mention some basic results that relate to our general discussion of finite groups, without proofs. For more details see, for example, Ref. [1].

Recall that the **permutation group** S_n is the set of all bijective maps of n objects, so

$$S_n = \{\sigma : \{1, ..., n\} \to \{1, ..., n\} \,|\, \sigma \text{ is a bijection}\} \ .$$

It is a finite group of order $|S_n| = n!$ with group multiplication given by composition of maps. The **sign** of permutations is a map $\mathrm{sgn} : S_n \to \{\pm 1\} \cong \mathbb{Z}_2$ which can be defined as

$$\mathrm{sgn}(\sigma) = \prod_{i<j} \frac{\sigma(j) - \sigma(i)}{j - i}$$

It is not too hard to check from this definition that $\mathrm{sgn}(\sigma_1 \circ \sigma_2) = \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)$, so that sgn is a group homomorphism - and indeed a non-trivial one-dimensional representation of S_n. Permutations $\sigma \in S_n$ with $\mathrm{sgn}(\sigma) = +1$ ($\mathrm{sgn}(\sigma) = -1$) are called **even permutations** (**odd permutations**). The kernel of sgn consists of all even permutations which, hence, form a group, called the **alternating group** A_n.

One way to write down a permutation σ is by explicitly listing its images, so

$$\sigma = \left[\begin{array}{cccc} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{array} \right] \ .$$

All permutations can be written in terms of cycles, that is, subsets of $\{1, \ldots, n\}$ which are cyclically permuted. For example, the following permutation $\sigma \in S_9$ reads in cycle notation

$$\sigma = \left[\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 1 & 6 & 5 & 8 & 7 & 9 \end{array} \right] \ , \qquad \sigma = (1, 2, 3, 4)(5, 6)(7, 8)(9) \ . \tag{2.19}$$

To discuss representations of S_n we need to know about its conjugacy classes. In fact, conjugacy classes of S_n consist of all permutations with the same length of cycles. We can label these lengths by **partitions** of n, that is, by integer vectors $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and $n = \lambda_1 + \cdots + \lambda_k$. In this language, the permutation $\sigma \in S_9$ in Eq. (2.19) is characterised by the partition of 9 given by $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4, 2, 2, 1)$. So, in short, conjugacy classes of S_n are labelled by partitions of n. Partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n can also be represented by **Young tableaux** which consist of n boxes arranged in k rows, with λ_i boxes in row i and the length of rows not increasing from top to bottom. For example, the Young tableau for the conjugacy class of S_9 which contains the permutation σ in Eq. (2.19) is

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array}$$

So, the conjugacy classes of S_n can be labelled by all the Young tableaux with n boxes. For a given Young tableau $\lambda = (\lambda_1, \ldots, \lambda_k)$ we can define the following subsets of S_n.

$$R_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves each row}\} \;, \qquad C_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves each column}\}$$

These are the subsets of permutations which permute the entries in each row (column) of the Young tableau such that they stay in the same row (column). Based on these two sets we can define a map $P_\lambda : A_{S_n} \to A_{S_n}$ by

$$P_\lambda = c \left(\sum_{\sigma \in R_\lambda} \sigma \right) \times \left(\sum_{\sigma \in C_\lambda} \mathrm{sgn}(\sigma)\sigma \right) \;,$$

where $c \in \mathbb{C}$. It can be shows that, for a suitable choice of c, this map is a projector, so $P_\lambda^2 = P_\lambda$. The irreps of S_n are in one-to-one correspondence with partitions λ so can be written as R_λ. Each subspace $P_\lambda A_{S_n}$ of the group algebra contains precisely one irrep R_λ. Note that this is actually a stronger statement than the one in Theorem 2.9 which only provided a projector onto a subspace which carries $\dim(R)$ copies of an irrep R.

Chapter 3

Lie groups

The remainder of the course is devoted to *Lie groups*, their associated *Lie algebras* and their representations. These continuous groups have wide-ranging applications in physics, and they describe most of the symmetries underlying the fundamental theories of physics, such as, for example, the standard model of particle physics. Examples of Lie groups include the (special) orthogonal groups $\mathrm{SO}(n)$, the (special) unitary groups $\mathrm{SU}(n)$ and other, less familiar groups of matrices. Many of these groups will be explicitly discussed later but first we should develop the general understanding of Lie groups. The obvious place to start is with their definition.

3.1 The geometry of Lie groups

Definition 3.1. *(Lie group) A group G is a* ***Lie group*** *if G is a differentiable manifold and the group multiplication and inversion are differentiable maps.*

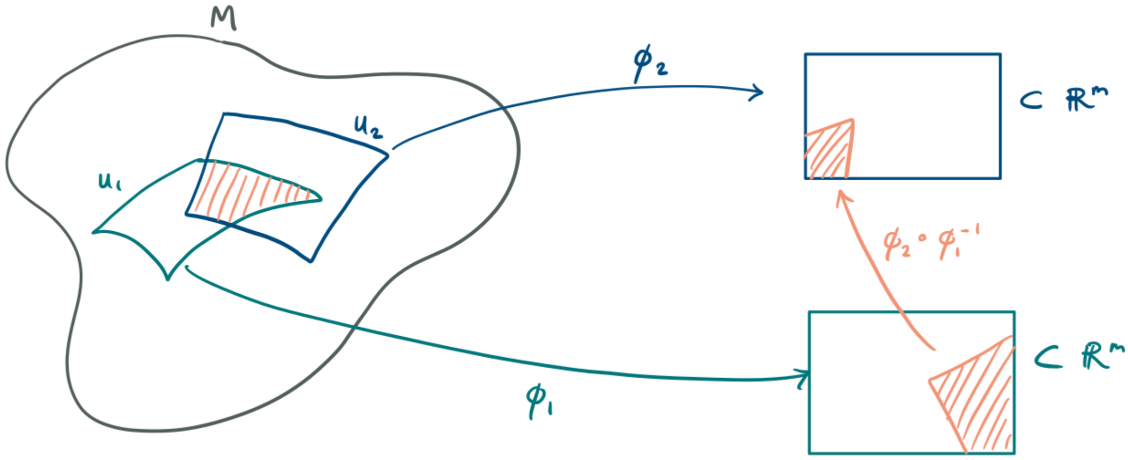
Very little can be extracted from this definition unless we know about differential manifolds. This is of course the subject of a different mathematical field - differential geometry - which we cannot possibly develop in any detail here. Instead, we will go for a short crash course (and ask mathematicians to suspend, for the time being, their sense of good mathematical taste).

Some (very) basic differential geometry

Definition of manifolds

We begin by defining differentiable manifolds.

Definition 3.2. *(Differentiable manifold) A* ***differentiable manifold*** *M is a topological space with* ***charts*** *(U_i, ϕ_i), where (U_i) is an open cover of M and $\phi_i : U_i \to V_i$ are homeomorphisms (they are continuous and have an inverse which is continuous) into the open sets $V_i \subset \mathbb{R}^n$. Further, the* ***transition functions*** *$\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are required to be $\mathcal{C}^\infty$ functions. The collection $(U_i, \phi_i)_i$ of all charts is called an* ***atlas*** *of M. The* ***dimension*** *of M is defined by $\dim(M) = n$.*



The intuitive meaning of this definition is indicated in the figure above. Every chart identifies a "patch" of the manifold with an open set in $\mathbb{R}^n$ and for this reason a manifold can be thought of as looking locally like $\mathbb{R}^n$. The entire manifold is covered in this way by charts which, together, form an atlas. Some open sets U_i and U_j will necessarily overlap and their images under the maps ϕ_i and ϕ_j are, in general, two different open sets in $\mathbb{R}^n$ which represent the same parts of the manifold (the regions shaded in red in the above figure). The transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ identify those open $\mathbb{R}^n$ sets which correspond to the same part of M and they are required to be differentiable (a meaningful requirement for maps on $\mathbb{R}^n$).

Vector fields

A (infinitely many times) differentiable real function $f : M \to \mathbb{R}$ on a manifold is a function such that all "local" functions $f \circ \phi_i^{-1} : V_i \to \mathbb{R}$ are (infinitely many times) differentiable. Since the transition functions are $\mathcal{C}^\infty$ this definition is consistent in view of the patch overlaps. The space of such functions is denoted by $\mathcal{C}^\infty(M)$.

Definition 3.3. *A* ***vector field*** *on a manifold M is a linear map $\xi : C^\infty(M) \to C^\infty(M)$ which satisfies the product rule*

$$\xi(fg) = f\xi(g) + \xi(f)g.$$

We define $\xi_x(f) = \xi(f)(x)$ for $x \in M$

This definition is inspired by $\mathbb{R}^n$ where a vector can be used to perform a partial derivative in the direction of the vector. Of course such partial derivatives are linear and they obey the product rule for differentiation. The above definition uses these properties as a basis for a more abstract definition.

The tangent space and tangent map

Definition 3.4. *The* ***tangent space*** *at a point x of a manifold M is defined by*

$$T_xM = \{\xi_x \mid \xi \text{ is a (local) vector field}\} .$$

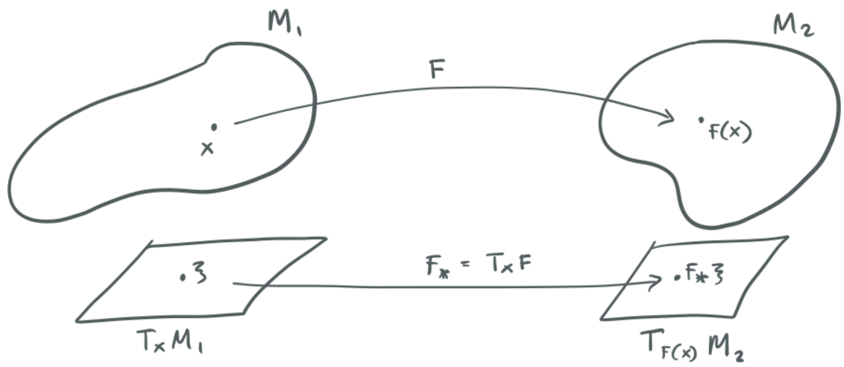
The tangent space is a vector space under the obvious addition and scalar multiplication, defined by $(\xi_x + \tilde{\xi}_x)(f) = \xi(f)(x) + \tilde{\xi}(f)(x)$ and $(\alpha\xi_x)(f) = \alpha\xi(f)(x)$, for two vector fields ξ and $\tilde{\xi}$.

Maps $F : M_1 \to M_2$ between two manifolds are called differentiable if their local versions $\phi_{2i} \circ F \circ \phi_{1j}^{-1} : V_{1i} \to V_{2j}$, relative to chart (ϕ_{1i}, U_{1i}) of M_1 and (ϕ_{2i}, U_{2i}) of M_2, are differentiable. Such differentiable maps between manifolds induce maps between the tangent spaces.

Definition 3.5. *For a differentiable map $F : M_1 \to M_2$ between manifolds M_1 and M_2, the* ***tangent map*** *$T_xF : T_xM_1 \to T_{F(x)}M_2$ is defined by*

$$T_xF(v)(f) \equiv v(f \circ F) \tag{3.1}$$

where $x \in M_1$, $v \in T_xM_1$ and $f \in \mathcal{C}^\infty(M_2)$. The tangent map is also denoted by F_*, so that, for a vector field ξ on M_1, $F_*\xi$ is a vector field on M_2 defined by $(F_*\xi)_{F(x)} = T_xF(\xi_x)$.



So the tangent map maps vector fields on M_1 into vector fields on M_2, in a way that is consistent with the structure of tangent spaces: if $M_1 \ni x \mapsto F(x) \in M_2$ then T_xF maps tangent vectors in T_xM_1 to those in $T_{F(x)}M_2$. All this is indicated in the above figure.

Remark 3.1. *(i) Consider two differentiable maps $G : M_1 \to M_2$ and $F : M_2 \to M_3$. Then, we have the chain rule*

$$T_x(F \circ G) = T_{G(x)}(F)T_x(G)\ . \tag{3.2}$$

This can be seen, using the definition (3.1) of the tangent map. Let $f \in \mathcal{C}^\infty(M_3)$, $v \in T_xM_1$ and $w = T_xG(v)$, then

$$\begin{aligned} T_x(F \circ G)(v)(f) &= v(f \circ F \circ G) = T_x(G)(v)(f \circ F) = w(f \circ F) \\ &= T_{G(x)}(F)(w)(f) = T_{G(x)}(F)T_x(G)(v)(f)\ . \end{aligned} \tag{3.3}$$

Alternative, the chain rule can be written as $(F \circ G)_* = F_*G_*$.

(ii) The tangent map of the identity map id_M is $T_x\mathrm{id}_M = \mathrm{id}_{T_xM}$, the identity map on the tangent spaces.

(iii) For an invertible differentiable map $F : M_1 \to M_2$, we can combine (i) and (ii) to get $(T_xF)^{-1} = T_{F(x)}(F^{-1})$.

It is useful to look at some of this in local coordinates. Consider a chart (ϕ, U) of M. With coordinates $(x^1, \ldots, x^n)$ on $\mathbb{R}^n$ (which is an example of a manifold), the tangent map $T_x\phi : T_xM \to T_{\phi(x)}V = T_{\phi(x)}\mathbb{R}^n$ associated to the chart map ϕ can be written as

$$T_x\phi(v) = v^i(x)\frac{\partial}{\partial x^i} .$$

This provides an explicit local identification of tangent vectors v with first order derivative operators. Since the map ϕ is invertible, so is $T_x\phi$ and this implies $\dim(T_xM) = \dim(T_{\phi(x)}\mathbb{R}^n) = n$. Hence,

$$\dim(T_xM) = \dim(M) , \tag{3.4}$$

that is, the dimension of the tangent space (as a vector space) equals the dimension of the manifold.

Tangent map in coordinates

Further, consider a differentiable map $F : M \to N$, a chart (ϕ, U) with $\phi(x) = (x^1, \ldots, x^n)$ on M and a chart (ψ, W) with $\psi(y) = (y^1, \ldots, y^m)$ on N. The chart maps ϕ and ψ both have associated tangent maps which map tangent vectors on M or N to tangent vector in $\mathbb{R}^n$. These images of tangent vectors can be written as

$$T_x\phi(v) = v^i(x)\frac{\partial}{\partial x^i} \in T_{\phi(x)}\mathbb{R}^n , \qquad T_y\psi(w) = w^i(y)\frac{\partial}{\partial y^i} \in T_{\psi(y)}\mathbb{R}^m ,$$

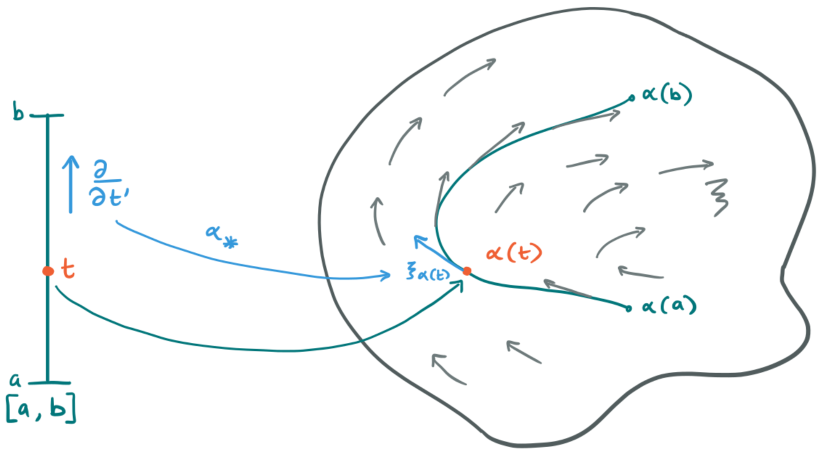
where $v^i(x)$ and $w^i(y)$ can be viewed as the coordinates of v and w, relative to the two charts. The local version of F relative to the chosen charts is $\mathcal{F} \equiv \psi \circ F \circ \phi^{-1}$ and it is instructive to work out its tangent map (setting $y = \mathcal{F}(x)$)

$$T_{\phi(x)}(\mathcal{F})\left(v^i\partial_{x^i}\right)(f)(x) = v^i\partial_{x^i}(f \circ \mathcal{F})(x) = v^i(\partial_{x_i}\mathcal{F}^j)(x)(\partial_{y_j}f)(y) = [D\mathcal{F}(x)]_i^j\, v^i\partial_{y_j}f(y) .$$

The result shows that the local version of the tangent map is described, relative to the basis of standard partial differentials ∂_{x_i} and ∂_{y_j}, by the Jacobi matrix $D\mathcal{F}$.

Integral curves and flows

A vector field has associated integral curves which are the curves whose tangent vectors equal the value of the vector field at every point along the curve. This is illustrated in the figure below.



The formal definition of integral curves is as follows.

Definition 3.6. *Let ξ be a vector field on M. An* ***integral curve*** *of ξ is a differentiable curve $\alpha_\xi : [a,b] \to M, t \in [a,b] \mapsto \alpha_\xi(t) \in M$ such that*

$$\partial_t \alpha_\xi(t) := T_t(\alpha_\xi)\,(\partial_t) = \xi_{\alpha(t)} \tag{3.5}$$

The left hand-side of Eq. (3.5) is of course symbolic - the partial derivative $\partial_t = \partial/\partial t$ does not really have a meaning on an abstract manifold - and is defined by the expression in the middle. In this context, the partial derivative ∂_t should be seen as a tangent vector on the interval $[a,b]$ (seen as a one-dimensional manifold with coordinate t) which is mapped to the tangent vector $T_t(\alpha_\xi)\,(\partial_t)$ to the curve. Eq. (3.5) demands that this tangent vector equals the value $\xi_{\alpha(t)}$ at that point and it is a first order ordinary differential equation for α_ξ. Provided an initial condition, such as $\alpha_\xi(0) = x \in M$, is specified it has a unique solution. A **flow** combines all these solutions for different initial conditions and is defined as follows.

Definition 3.7. *The* ***flow*** *$\phi(t,x) = \phi_t(x)$ of the vector field ξ on M is given by the unique integral curve with the initial condition $\phi(0,x) = x$.*

Remark 3.2. *(i) The maps $s \mapsto \phi(s,\phi(t,x))$ and $s \mapsto \phi(s+t,x)$ have the same initial value $\phi(t,x)$ at $s = 0$ and the same derivative with respect to s so by uniqueness of the solution they must be equal. This implies*

$$\phi(s,\phi(t,x)) = \phi(s+t,x) \quad \Longrightarrow \quad \phi_s \circ \phi_t = \phi_{s+t} \tag{3.6}$$

(ii) The underlying vector field ξ can be recovered from its flow by $\partial_t|_0 \phi(t,x) = \xi_x$.

This concludes our very rudimentary account of basic differential geometry.

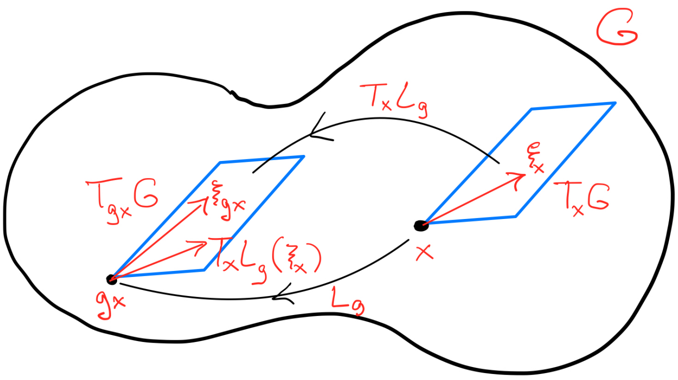
We now return to Lie groups. Recall that, in general, a group representation is a group homomorphism $R : G \to \mathrm{GL}(V)$. Since $\mathrm{GL}(V)$ can be identified with $n \times n$ matrices $M \in \mathrm{GL}(\mathbb{F}^n)$ with $\det(M) \neq 0$ (and this is an open condition), general linear groups are Lie groups. Hence, we should think of representations of Lie groups G as Lie group morphisms $R : G \to \mathrm{GL}(V)$ into the specific Lie group $\mathrm{GL}(V)$, where Lie group morphisms means that it has to respect the differential manifold structure, that is, be a differential map, and the group structure, that is, be a group homomorphism.

Left-invariant vector fields

The interplay between group and manifold structures leads to a number of new features which we now explore. For a start, the existence of the group structure means that Lie groups G have special differentiable maps $L_g : G \to G$, called **left translations**, defined in terms of the group multiplication by $L_g(x) = gx$. The tangent maps of these left translations can be used to single out specific vector fields on G.

Definition 3.8. *A vector field ξ on G is called* ***left-invariant*** *if $T_x L_g(\xi_x) = \xi_{gx}$ for all $x, g \in G$. (Equivalently, this can also be written as $g_*\xi = \xi$.)*

This definition might look confusing at first but has, in fact, a very intuitive interpretation which is illustrated in the figure below.



On the group manifold G a vector field ξ provides tangent vectors at every point on the manifold, so in particular at $\xi_x \in T_xG$ and $\xi_{gx} \in T_{gx}G$. Another way to obtain a tangent vector in $T_{gx}G$ is to act on ξ_x with the tangent map $T_xL_g : T_xG \to T_{gx}G$ associated to the left-translation L_g. This gives a vector $T_xL_g(\xi_x) \in T_{gx}G$ which is, in general, different from the value ξ_{gx} of the vector field at gx. The vector field is called left-invariant iff these two vectors are equal for all $x, g \in G$.

We denote the space of left-invariant vector fields on G by $\mathcal{L}(G)$ and, since the left-invariance condition is linear, this is clearly a vector space. The left-invariance condition means that a left-invariant vector field ξ is uniquely determined once we know its value ξ_x at one (any) point on G and conversely fixing a vector $v \in T_xG$ there is a left-invariant vector field with $\xi_x = v$. This means we have a vector space isomorphism

$$\begin{array}{ccc} \mathcal{L}(G) & \stackrel{\cong}{\to} & T_eG \\ \xi & \mapsto & \xi_e \end{array} \tag{3.7}$$

between the space of left-invariant vector fields and the tangent space at the identity (or any other tangent space for that matter). The inverse of this map is given by $\xi_e \mapsto g_*\xi_e = \xi_g$. In particular, the isomorphism (3.7) implies that the (vector space) dimension of $\mathcal{L}(G)$ is the same as the tangent space dimension and, hence, from Eq. (3.4) as same as the dimension of G as a manifold.

$$\dim(\mathcal{L}(G)) = \dim(T_eG) = \dim(G)\ . \tag{3.8}$$

The bracket for left-invariant vector fields

Since vector fields ξ, η are derivative operators we can consider their commutator $[\xi, \eta] := \xi \circ \eta - \eta \circ \xi$. The following quick calculation in coordinates

$$[\xi, \eta] = \xi^i \partial_{x^i} \circ \eta^j \partial_{x^j} - \eta^i \partial_{x^i} \circ \xi^j \partial_{x^j} = \left(\xi^i \partial_{x^i} \eta^j - \eta^i \partial_{x^i} \xi^j\right) \partial_{x^j}$$

shows that, perhaps contrary to naive expectation, the commutator is still a first order derivative operator, so it is a vector field as well. What is more, the property of left-invariance is preserved by the commutator.

Lemma 3.1. *If ξ, η are left-invariant vector fields on G, then so is $[\xi, \eta]$.*

Proof. All the proof requires is a careful calculation, based on the various definitions. We start with two left-invariant vectors fields ξ and η, so vector fields satisfying $g_*^{-1}\xi = \xi$ and $g_*^{-1}\eta = \eta$. It is useful to first work out the left-hand sides of these conditions acting on a function $f : G \to \mathbb{R}$ which gives

$$g_*^{-1}\xi_x(f) = \xi_{gx}(f \circ g^{-1}) = \xi(f \circ g^{-1})(gx) \quad \Rightarrow \quad g_*^{-1}\xi(f) = \xi(f \circ g^{-1}) \circ g \ ,$$

and similarly for η. Using this equation we re-write the commutator as

$$\begin{aligned}[g_*^{-1}\xi, g_*^{-1}\eta](f) &= g_*^{-1}\xi\left(g_*^{-1}\eta(f)\right) - g_*^{-1}\eta\left(g_*^{-1}\xi(f)\right) \\ &= g_*^{-1}\xi\left(\eta(f \circ g^{-1}) \circ g\right) - g_*^{-1}\eta\left(\xi(f \circ g^{-1}) \circ g\right) \\ &= \xi\left(\eta(f \circ g^{-1})\right) \circ g - \eta\left(\xi(f \circ g^{-1})\right) \circ g = [\xi, \eta](f \circ g^{-1}) \circ g = g_*^{-1}[\xi, \eta](f) \ .\end{aligned}$$

Hence, g_*^{-1} can be "pulled in and out" of the commutator and this means $g_*^{-1}[\xi, \eta] = [g_*^{-1}\xi, g_*^{-1}\eta] = [\xi, \eta]$ where the last equality follows from the left-invariance of ξ, η. □

3.2 Lie algebras

The previous Lemma shows that the left-invariant vector fields $\mathcal{L}(G)$ do not just form a vector space but also carry a commutator. It is useful to define such a structure more abstractly.

Definition and Lie algebras of a Lie group

Definition 3.9. *(Lie algebra) A* ***Lie algebra*** $\mathcal{L}$ *is a vector space endowed with a bilinear bracket operation (called the* ***Lie bracket*** *or sometimes the commutator)* $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ *satisfying*

(i) $[\xi, \eta] = -[\eta, \xi]$ *(anti-symmetry)*

(ii) $[\xi, [\eta, \lambda]] + [\eta, [\lambda, \xi]] + [\lambda, [\xi, \eta]] = 0$ *(Jacobi identity)*

A Lie algebra morphism $f : \mathcal{L}_1 \to \mathcal{L}_2$ *between two Lie algebras* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *is a linear map which preserves the commutator, so* $f([\xi, \eta]) = [f(\xi), f(\eta)]$ *for all* $\xi, \eta \in \mathcal{L}_1$*.*

The commutator $[\xi, \eta] := \xi \circ \eta - \eta \circ \xi$ of two vector fields is evidently anti-symmetric and a quick calculation shows that it also automatically satisfies the Jacobi identity.

Exercise 3.1. *Show that the vector field commutator* $[\xi, \eta] := \xi \circ \eta - \eta \circ \xi$ *satisfies the Jacobi identity.*

Further, from Lemma 3.1 the left-invariant vector fields $\mathcal{L}(G)$ on G are closed under the commutator and we conclude that $\mathcal{L}(G)$ is, in fact, a Lie algebra, also called the **Lie algebra associated to the group** G.

Representations of Lie algebras

We have defined representations of a Lie group G as group homomorphisms into the specific Lie groups $\mathrm{GL}(V)$. The Lie algebra of $\mathrm{GL}(V)$ is given by $\mathrm{End}(V)$, the endomorphisms or linear maps $V \to V$. (To see this note that $\mathrm{GL}(V) \cong \mathrm{GL}(\mathbb{F}^n)$ which is locally like $\mathbb{F}^{n^2}$ and, hence, has a tangent space $\mathbb{F}^{n^2}$). Hence, it makes sense to define representations of Lie algebras as Lie algebra morphisms into the specific Lie algebras $\mathrm{End}(V)$.

Definition 3.10. *(Representation of a Lie algebra) A* ***representation*** *of a Lie algebra $\mathcal{L}$ is a linear map $r : \mathcal{L} \to \mathrm{End}(V)$ such that*

$$r([\xi,\eta]) = [r(\xi), r(\eta)]. \tag{3.9}$$

for all $\xi, \eta \in \mathcal{L}$.

We can define reducible, irreducible and fully reducible representations of Lie algebras in complete analogy with the corresponding definitions for (Lie) groups.

Basis on a Lie algebra

In physics applications, it is common to introduce a basis $(\xi_1, \ldots, \xi_n)$ on a Lie algebra $\mathcal{L}$. The commutator $[\xi_i, \xi_j] \in \mathcal{L}$ must be a linear combination of these basis vectors, so we can write

$$[\xi_i, \xi_j] = f_{ij}{}^k \xi_k \ . \tag{3.10}$$

The constants $f_{ij}{}^k$ which appear in these equations are called the **structure constants** of the Lie algebra $\mathcal{L}$. Relative to such a basis, the condition (3.9) for a linear map $r : \mathcal{L} \to \mathrm{End}(V)$ to be a Lie algebra representation is equivalent to

$$[r(\xi_i), r(\xi_j)] = f_{ij}{}^k r(\xi_k) \ ,$$

for all $i, j = 1, \ldots, n$. This simply means that the Lie algebra basis ξ_i and their representation maps $r(\xi_i)$ have to commute on the same structure constants $f_{ij}{}^k$ and it points to a recipe for how to find Lie algebra representations: identify linear maps (or matrices) which commute "in the same way" as the basis elements of the Lie algebra.

3.3 The adjoint representation

Now that we have a vector space - the Lie algebra - associated to a Lie group it is natural to ask if we can represent the Lie group (and the Lie algebra) on this vector space. This leads to a specific and important representation called the **adjoint representation**.

Adjoint at the group level

As a preparation, we introduce the **conjugation map**

$$C_g : G \to G \ , \qquad C_g(x) = gxg^{-1} \ ,$$

which acts simply by conjugating its argument with $g \in G$. Such a conjugation with a fixed group element defines an automorphism $C_g \in \mathrm{Aut}(G)$ (where $\mathrm{Aut}(G)$ is the group which consists of all invertible group homomorphism $G \to G$). This means we also have a group homomorphism

$$C : G \to \mathrm{Aut}(G) \ , \qquad g \mapsto C_g \ ,$$

which assigns to a group element g the associated conjugation automorphism C_g. We can check that C is indeed a group homomorphism by

$$C_{g_1g_2}(x) = g_1g_2x(g_1g_2)^{-1} = g_1(g_2xg_2^{-1})g_1^{-1} = C_{g_1} \circ C_{g_2}(x) \quad \Rightarrow \quad C_{g_1g_2} = C_{g_1} \circ C_{g_2} \ . \tag{3.11}$$

Note that $C_g(e) = e$ - the group identity is invariant under conjugation - so that the tangent map $T_eC_g : T_eG \to T_eG$ is a map from the tangent space at e to itself. This observation facilitates the following definition.

Definition 3.11. *(Adjoint representation of a Lie group) The* ***adjoint representation*** *of a Lie group G is a representation* $\mathrm{Ad}: G \to \mathrm{GL}(\mathcal{L}(G))$ *of the Lie group on its own Lie algebra* $\mathcal{L}(G) \cong T_eG$ *defined by*

$$\mathrm{Ad}(g) = T_eC_g : T_eG \to T_eG \tag{3.12}$$

We should convince ourselves that Ad is indeed a representation:

$$\mathrm{Ad}(g_1g_2) = T_e(C_{g_1g_2}) \stackrel{(3.11)}{=} T_e(C_{g_1} \circ C_{g_2}) \stackrel{(3.3)}{=} T_eC_{g_1} \circ T_eC_{g_2} = \mathrm{Ad}(g_1) \circ \mathrm{Ad}(g_2) \ .$$

Adjoint at the algebra level

In order to define a corresponding representation of the Lie algebra $\mathcal{L}(G)$ we can use the tangent map $T_e\mathrm{Ad}$ of the adjoint representation. Since $\mathrm{Ad}(e) = \mathrm{id}$ it maps the Lie-algebra $\mathcal{L}(G) \cong T_eG$ of G into the Lie-algebra $\mathrm{End}(\mathcal{L}(G))$ of $\mathrm{GL}(\mathcal{L}(G))$, so that the following definition makes sense.

Definition 3.12. *(Adjoint representation of Lie algebra) The* ***adjoint representation*** $\mathrm{ad}: \mathcal{L}(G) \to \mathrm{End}(\mathcal{L}(G))$ *of the Lie algebra* $\mathcal{L}(G)$ *on itself is defined by* $\mathrm{ad} = T_e\mathrm{Ad}$.

While it was quite straightforward to show that Ad is a group representation, showing that ad is a Lie algebra representation is a bit more involved. To do this, we have to think about flows.

For a vector field ξ on Lie group G with flow ϕ we can think about the specific integral curve

$$\alpha_\xi(t) = \phi(t, e) \tag{3.13}$$

which passed through the identity, that is, $\alpha_\xi(0) = e$. If ξ is a left-invariant vector field then the entire flow can be reconstructed from this curve by

$$\phi(t, x) = x\,\alpha_\xi(t) \ . \tag{3.14}$$

This follows because the two sides of Eq. (3.14) satisfy the same initial condition, $\phi(0, x) = x$ and $x\alpha(0) = xe = x$, and because the right-hand side satisfies the same differential equation as the flow ϕ:

$$\partial_t|_0(x\alpha_\xi(t) = T_0(L_x \circ \alpha_\xi)(\partial_t) \stackrel{(3.3)}{=} T_{\alpha_\xi(0)}L_x \circ T_0\alpha_\xi(\partial_t) = T_eL_x(\xi_e) = \xi_x \ ,$$

where left-invariance of ξ has been used in the last step. Further, the integral curve α_ξ satisfies the nice property

$$\alpha_\xi(s + t) = \alpha_\xi(s)\alpha_\xi(t) \tag{3.15}$$

which follows from

$$\alpha_\xi(s + t) \stackrel{(3.13)}{=} \phi(s + t, e) \stackrel{(3.6)}{=} \phi(t, \phi_s(e)) \stackrel{(3.14)}{=} \phi_s(e)\alpha_\xi(t) \stackrel{(3.13)}{=} \alpha_\xi(s)\alpha_\xi(t) \ .$$

We are now ready to prove the crucial theorem which expressed the ad representation in terms of the commutator.

Theorem 3.2. $\mathrm{ad}(\xi_e)(\eta_e) = [\xi, \eta]_e$

Proof. We start with two left-invariant vector fields ξ and η on G with associated curves α_ξ and α_η, defined as in Eq. (3.13). The goal is to work out the expression $\mathrm{ad}(\xi_e)(\eta_e)$ step by step, following the definition of Ad and ad. We start with the conjugation map.

$$C_{\alpha_\xi(s)}(\alpha_\eta(t)) = \alpha_\xi(s)\cdot\alpha_\eta(t)\cdot\alpha_\xi(s)^{-1} = \alpha_\xi(s)\cdot\alpha_\eta(t)\circ\alpha_\xi(-s)\ .$$

For the group adjoint this implies

$$\mathrm{Ad}(\alpha_\xi(s))\eta_e = T_e(C_{\alpha_\xi(s)})\eta_e = T_e(C_{\alpha_\xi(s)})T_0(\alpha_\eta)\,(\partial_t) = T_0(C_{\alpha_\xi(s)}\circ\alpha_\eta)\,(\partial_t) = \partial_t|_0\alpha_\xi(s)\cdot\alpha_\eta(t)\cdot\alpha_\xi(-s)$$

which translates into

$$\begin{aligned}\mathrm{ad}(\xi_e)(\eta_e) &= T_e\mathrm{Ad}(\xi_e)(\eta_e) = T_e\mathrm{Ad}\;T_0\alpha_\xi\,(\partial_s)\,(\eta_e) = T_0(\mathrm{Ad}\circ\alpha_\xi)\,(\partial_s)\,\eta_e = \partial_s|_0\,\mathrm{Ad}(\alpha_\xi(s))\eta_e\\ &= \partial_s\partial_t|_0\,\alpha_\xi(s)\cdot\alpha_\eta(t)\cdot\alpha_\xi(-s)\end{aligned}$$

and finally

$$\begin{aligned}\mathrm{ad}(\xi_e)(\eta_e)(f) &= \partial_s\partial_t|_0\,f(\alpha_\xi(s)\cdot\alpha_\eta(t)\cdot\alpha_\xi(-s))\\ &= \partial_s\partial_t|_0\,f(\alpha_\xi(s)\cdot\alpha_\eta(t)) - \partial_s\partial_t|_0\,f(\alpha_\eta(t)\cdot\alpha_\xi(s)) = \xi_e\eta_e(f) - \eta_e\xi_e(f)\ .\end{aligned}$$

□

Corollary 3.1. *The **adjoint representation** $\mathrm{ad}:\mathcal{L}(G)\to\mathrm{End}(\mathcal{L}(G))$ is a Lie algebra representation.*

Proof. This follows by combining the previous theorem with the Jacobi identity.

$$\begin{aligned}\mathrm{ad}([\xi,\eta])(\lambda) &\overset{\text{Thm. (3.2)}}{=} [[\xi,\eta],\lambda] \overset{\text{Jacobi}}{=} [\xi,[\eta,\lambda]]-[\eta,[\xi,\lambda]]\\ &= \mathrm{ad}(\xi)\circ\mathrm{ad}(\eta)(\lambda)-\mathrm{ad}(\eta)\circ\mathrm{ad}(\xi)(\lambda) = [\mathrm{ad}(\xi),\mathrm{ad}(\eta)](\lambda)\end{aligned}$$

□

Adjoint representation in a basis

Suppose we choose a basis $(\xi_1,\dots,\xi_n)$ on $\mathcal{L}(G)$ and we want to work out the representation matrices of ad relative to this basis. From the previous corollary and Eq. (3.10) it follows that

$$\mathrm{ad}(\xi_i)(\xi_j) = [\xi_i,\xi_j] = f_{ij}{}^k\xi_k \tag{3.16}$$

and, hence, that the representation matrices relative to this basis are given by the structure constants, so

$$[\mathrm{ad}(\xi_i)]^k_j = f_{ij}{}^k\ . \tag{3.17}$$

Relation between group and algebra

We have now introduced Lie groups and their Lie algebras and we understand - at least in principle - how to obtain the algebra from the group. For both the group and the algebra we have defined morphisms (representations) and we have just seen an example of a group representation, the adjoint, leading to a representation of its Lie algebra via taking the tangent map at the identity. All this suggests a close correspondence between Lie groups, their associated algebras and representations and leads to a set of questions.

(1) Is it always true that the tangent map at e of a group homomorphism (a representation) is a Lie algebra homomorphism (a representation of the Lie algebra)?

(2) Are all Lie algebra homomorphisms tangent maps at e of group homomorphisms?

(3) Can the Lie group be "recovered" from the Lie algebra?

Question (1) can be answered by carefully thinking about the various definitions involved and expressing them in terms of diagrams.

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[d, "C_g"'] & \tilde{G} \arrow[d, "C_{F(g)}"] \\ G \arrow[r, "F"'] & \tilde{G} \end{tikzcd} \qquad \begin{tikzcd} T_eG \arrow[r, "T_e(F)"] \arrow[d, "\mathrm{Ad}(g)"'] & T_{\tilde{e}}\tilde{G} \arrow[d, "\mathrm{Ad}(F(g))"] \\ T_eG \arrow[r, "T_e(F)"'] & T_{\tilde{e}}\tilde{G} \end{tikzcd}$$

These diagrams **commute** which means that either of the two paths from the upper left to the lower right leads to the same results. That this is the case can be checked by explicit calculation.

$$F \circ C_g(x) = F(gxg^{-1}) = F(g) \circ F(x) \circ F(g)^{-1} = C_{F(g)} \circ F(x) \tag{3.18}$$
$$T_e(F) \circ \mathrm{Ad}(g) = T_e(F) \circ T_e(C_g) = T_{\tilde{e}}(C_{F(g)}) \circ T_e(F) = \mathrm{Ad}(F(g)) \circ T_e(F) \tag{3.19}$$

A further, related diagram is

$$\begin{tikzcd} T_eG \arrow[r, "T_e(F)"] \arrow[d, "\mathrm{ad}(\xi)"'] & T_{\tilde{e}}\tilde{G} \arrow[d, "\mathrm{ad}(T_e(F)\xi)"] \\ \mathcal{L}(G) \cong T_eG \arrow[r, "T_e(F)"'] & T_{\tilde{e}}\tilde{G} \cong \mathcal{L}(\tilde{G}) \end{tikzcd}$$

whose commutativity

$$T_e(F) \circ \mathrm{ad}(\xi)(\eta) = \mathrm{ad} \circ T_e(F)(\xi) \circ T_e(F)(\eta) \quad \Rightarrow \quad T_e(F)([\xi, \eta]) = [T_e(F)\xi, T_e(F)\eta]$$

follows by taking the tangent map of Eq. (3.19) and it directly implies that $T_e(F)$ is a Lie algebra morphism. Conversely, we have the following theorem.

Theorem 3.3. *A linear map $f : \mathcal{L}(G) \to \mathcal{L}(\tilde{G})$ is the tangent map of a group homomorphism F (so $f = T_e(F) = F_*$) iff f is a Lie algebra homomorphism.*

Proof. See Ref. [1], p. 119. □

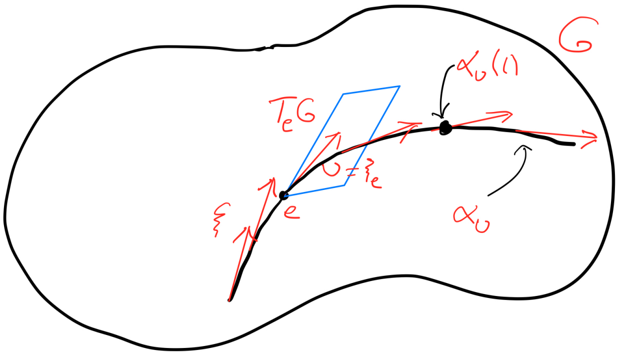
We have now answered questions (1) and (2) and in both cases the answer is "yes".

3.4 The exponential map

In order to address question (3) we need to discuss the **exponential map** which reconstructs the Lie group from its Lie algebra.

Definition 3.13. *For a left-invariant vector field ξ on G with $v = \xi_e$ we have an integral curve α_v with initial condition $\alpha_v(0) = e$. Then, the exponential map* $\mathrm{Exp} : T_eG \to G$ *is defined by* $\mathrm{Exp}(v) = \alpha_v(1)$*.*

Note that we have identified the tangent space at e with the Lie algebra, so Exp is indeed a map from the algebra into the group. Its geometrical interpretation is indicated in the figure below.



The exponential map has the following properties.

Theorem 3.4. *The exponential map*

(1) is differentiable at the origin and $T_0(Exp) = id_{\mathcal{L}(G)}$.
(2) maps $\mathcal{L}(G) \cong T_e(G)$ diffeomorphically into a neighbourhood of $e \in G$.
(3) satisfies $F \circ Exp = \widetilde{Exp} \circ T_e(F)$ for a group homomorphism $F : G \to \tilde{G}$.

Proof. (1) The fact that Exp is differentiable at the origin follows from general theorems on solutions to differential equations. For the second part of the statement we first note that the two integral curves $s \mapsto \alpha_{tv}(s)$ and $s \mapsto \alpha_v(ts)$ have the same tangent vector tv at e and must, hence, be equal. If we define the map m_v by $m_v(t) = tv$ we have

$$\mathrm{Exp} \circ m_v(t) = \mathrm{Exp}(vt) = \alpha_{vt}(1) = \alpha_v(t) \quad \overset{\partial_t|_0}{\Longrightarrow} \quad T_0\mathrm{Exp} \circ \underbrace{\dot{m}_v(0)}_{=v} = v$$

(2) Since $T_0\mathrm{Exp} = \mathrm{id}_{\mathcal{L}(G)}$, the exponential map is locally diffeomorphic by the inverse function theorem.

(3) If α_v is an integral curve for a left-invariant vector ξ with $\xi_e = v$ on G then $\beta_w = F \circ \alpha_v$ is an integral curve on $\tilde{G}$, associated to the left-invariant vector field $\tilde{\xi}$ with $\tilde{\xi}_{\tilde{e}} = w$. (Group homomorphisms map left-invariant vector fields to left-invariant vector fields and this is checked in the lemma below.) We have

$$w = \dot{\beta}_w(0) = T_eF\left(\dot{\alpha}_v(0)\right) = T_eF(v)$$

and, hence,

$$\widetilde{\mathrm{Exp}} \circ T_eF(v) = \widetilde{\mathrm{Exp}}(T_eF(v)) = \widetilde{\mathrm{Exp}}(w) = \beta_w(1) = F \circ \alpha_v(1) = F \circ \mathrm{Exp}(v)\ .$$

□

Lemma 3.2. *If $F : G \to \tilde{G}$ is a group homomorphism and ξ is a left-invariant vector field on G, then $\tilde{\xi} = F_*(\xi)$ is a left-invariant vector field on $\tilde{G}$.*

Proof. We start with $x, g \in G$, a left-invariant vector ξ on G and their images $\tilde{x} = F(x)$, $\tilde{g} = F(g)$ and $\tilde{\xi} = F_*(\xi)$ under F. As a first step, we express the left-translation $L_{\tilde{g}}$ on $\tilde{G}$ in terms of the left-translation L_g on G.

$$L_{\tilde{g}} \circ F(x) = L_{\tilde{g}}\tilde{x} = \tilde{g}\tilde{x} = F(g)F(x) = F(gx) = F \circ L_g(x) \quad \Rightarrow \quad L_{\tilde{g}} \circ F = F \circ L_g \ .$$

We need to check the left-invariance condition for $\tilde{x}$.

$$\begin{aligned} T_{\tilde{x}} L_{\tilde{g}}(\tilde{\xi}_{\tilde{x}}) &= T_{\tilde{x}} L_{\tilde{g}} \circ T_x F(\xi_x) = T_x(L_{\tilde{g}} \circ F)(\xi_x) = T_x(F \circ L_g)(\xi_x) \\ &= T_{gx} F \circ T_x L_g(\xi_x) = T_{gx} F(\xi_{gx}) = \tilde{\xi}_{F(gx)} = \tilde{\xi}_{\tilde{g}\tilde{x}} \end{aligned}$$

□

Our results mean that Lie groups can, at least locally, be recovered from the Lie algebra [1] This means we can consider classifying Lie groups by classifying their algebras. Moreover, Theorem 3.4 (3), which translates into the commutative diagram

$$\begin{tikzcd} G \arrow[r, "F"] & \tilde{G} \\ \mathcal{L}(G) \arrow[u, "\mathrm{Exp}"] \arrow[r, "T_e(F)"'] & \mathcal{L}(\tilde{G}) \arrow[u, "\widetilde{\mathrm{Exp}}"'] \end{tikzcd}$$

facilitates studying Lie group representations in terms of representations of the associated Lie algebras. We start with a Lie group G, work out its Lie-algebra $\mathcal{L}(G)$ and find a representation $r : \mathcal{L}(G) \to \mathrm{End}(V)$ of this Lie algebra. The corresponding Lie group representation $R : G \to \mathrm{GL}(V)$ with $T_e R = r$ can be obtained by exponentiating.

3.5 Matrix Lie groups

Many of the Lie groups we will consider are matrix Lie groups, that is, Lie groups $G \subset \mathrm{GL}(\mathbb{F}^d)$ which consist of non-singular $d \times d$ matrices with real or complex entries. For this reason and also in order to dial down the level of formality it makes sense to work out what we have developed so far for the case of matrix Lie groups.

Notation

To do this we work in a chart around the group identity $\mathbb{1}_d$ and parametrise the matrices in G as $g = g(t)$, where $t = (t^1, \ldots, t^n) \in \mathbb{R}^n$ are the parameters and we adjust conventions such that $g(0) = \mathbb{1}_d$. Further, we use $\mu, \nu, \ldots = 1, \ldots, d$ as matrix indices, so the $d \times d$ matrix g has entries $g_\mu{}^\nu$.

[1] In fact, there is a somewhat stronger result which states that the entire component of G connected to e is obtained by exponentiating the Lie algebra [1].

Vector fields

Vector fields ξ on the matrix Lie group can be written as

$$\xi_t = \xi^i(t)\frac{\partial}{\partial t^i} = \xi^i(t)\frac{\partial g_\mu{}^\nu}{\partial t^i}(t)\frac{\partial}{\partial g_\mu{}^\nu} = \xi^i(t)\,\mathrm{tr}\left(\frac{\partial g}{\partial t^i}(t)\frac{\partial}{\partial g^T}\right) , \tag{3.20}$$

where the last expression is just a short-hand for the one with indices in the middle.

Generators

The generators of G are $d \times d$ matrices defined by

$$T_i = \frac{\partial g}{\partial t^i}(0) , \tag{3.21}$$

so that we can expand the group matrices near the identity as $g(t) = \mathbb{1}_d + T_i t^i + \mathcal{O}(t^2)$.

Tangent space at identity

The tangent space at the identity consists of the value $\xi_{t=0}$ of vector fields which, from Eq. (3.20), can be written as

$$\xi_{t=0} = \xi^i(0)\,\mathrm{tr}\left(T_i\frac{\partial}{\partial g^T}\right) \tag{3.22}$$

so that the tangent space at the identity can be written as

$$T_{\mathbb{1}}G = \left\{v^i\,\mathrm{tr}\left(T_i\frac{\partial}{\partial g^T}\right) \,|\, v \in \mathbb{R}^n\right\} \cong \{v^i T_i \,|\, v \in \mathbb{R}^n\} \cong \mathcal{L}(G) . \tag{3.23}$$

The conclusion is that the tangent space at the identity (which is isomorphic to the Lie algebra from our general arguments) is the vector space of matrices spanned by the generators.

Left-invariant vector fields

Next, we would like to work out a more explicit form for the left-invariant vector fields on G. The left-translation, $L_g(x) = gx$ on g is just matrix multiplication, so we have

$$(gx)_\mu{}^\nu = g_\mu{}^\tau x_\tau{}^\nu \quad \Rightarrow \quad (T_x L_g)^{\nu\rho}_{\mu\sigma} = \frac{\partial (gx)_\mu{}^\nu}{\partial x_\rho{}^\sigma} = g_\mu{}^\tau \delta^\rho_\tau \delta^\nu_\sigma = g_\mu{}^\rho \delta^\nu_\sigma .$$

Inserting a vector field ξ in the form (3.22) into the left-invariance condition $\xi_{gx} = T_x L_g(\xi_x)$ gives

$$\xi^i(gt)\frac{\partial g_\mu{}^\nu}{\partial t^i}(gx)\frac{\partial}{\partial g_\mu{}^\nu} = (T_x L_g)^{\nu\rho}_{\mu\sigma}\xi^i(t)\frac{\partial g_\rho{}^\sigma}{\partial t^i}(x)\frac{\partial}{\partial g_\mu{}^\nu}$$

and replacing the tangent map by the above result, stripping off the differentials $\partial/\partial g_\mu{}^\nu$ and writing the result in matrix form gives

$$\xi^i(gt)\frac{\partial g}{\partial t^i}(gx) = \xi^i(t)g\frac{\partial g}{\partial t^i}(x) \quad \overset{x=\mathbb{1},t=0}{\Longrightarrow} \quad \xi^i\frac{\partial g}{\partial t^i} = \xi^i(0)gT_i .$$

We should think about the last equation as a differential equation which specifies the left-invariant vector field ξ, with components ξ^i whose value at $t = 0$ is $\xi^i(0)$. Denote by $L_i = \xi_i^j \partial_{t^j}$, where $i = 1, \ldots, n$, a basis of left-invariant vector fields, characterised by $\xi_i^j(0) = \delta_i^j$. These satisfy

$$\xi_i^j \frac{\partial g}{\partial t^j} = gT_i \ . \tag{3.24}$$

Hence, the left-invariant vector fields L_i can be written as

$$L_i = \xi_i^j \frac{\partial}{\partial t^j} = \xi_i^j \mathrm{tr}\left(\frac{\partial g}{\partial t^j}\frac{\partial}{\partial g^T}\right) = \mathrm{tr}\left(gT_i \frac{\partial}{\partial g^T}\right) \quad \Rightarrow \quad L_{i,\mathbb{1}} = \mathrm{tr}\left(T_i \frac{\partial}{\partial g^T}\right)$$

The last expression shows that the identification $\mathcal{L}(G) \cong T_{\mathbb{1}}G$ is explicitly given by $L_i \mapsto T_i$. In effect, this means we can think of the Lie algebra as the span of the generators - a much more concrete realisation than the one by left-invariant vector fields. What remains is to translate a number of operations which we have formulated in terms of left-invariant vector fields into the language of generators.

Commutator

The first of these is the commutator.

$$[L_i, L_j] = \mathrm{tr}\left(gT_i \frac{\partial}{\partial g^T}\right) \circ \mathrm{tr}\left(gT_j \frac{\partial}{\partial g^T}\right) - (i \leftrightarrow j) = \mathrm{tr}\left(g[T_i, T_j] \frac{\partial}{\partial g^T}\right) \in \mathcal{L}(G) \tag{3.25}$$

Here, the expression $[T_i, T_j]$ is the matrix commutator and since the right-hand side of Eq. (3.25) must be in the Lie algebra it follows that $[T_i, T_j]$ must be a linear combination of the generators. Hence, we can write

$$[T_i, T_j] = f_{ij}{}^k T_k \tag{3.26}$$

for structure constants $f_{ij}{}^k$ and inserting this back into Eq. (3.25) shows that these are, in fact, the same as the structure constants for L_i, so $[L_i, L_j] = f_{ij}{}^k L_k$. The upshot of this discussion is that, instead of having to work with left-invariant vector fields (first order derivative operators) and their commutators we can instead work with the generators and the matrix commutator.

Exponential map

The next step is to translate the exponential map into the language of generators. Consider a left-invariant vector field $\xi^i = v^j \xi_j^i$ and define the associated generator $T = v^i T_i$. The integral curve $t^i = t^i(s)$ and $\alpha_v(s) = g(t(s))$ of this left-invariant vector field satisfy the differential equation

$$\frac{dt^i}{ds} = v^j \xi_j^i \quad \Rightarrow \quad \frac{d\alpha_v}{ds} = \frac{\partial g}{\partial t^i}\frac{dt^i}{ds} = \frac{\partial g}{\partial t^i} v^j \xi_j^i \overset{(3.24)}{=} v^j g T_j = \alpha_v(s) T$$

with initial condition $\alpha_v(0) = g(t(0)) = g(0) = \mathbb{1}$. The solution is simply $\alpha_v(s) = \exp(sT)$, where exp is the matrix exponential. Hence, the exponential map in terms of generator is

$$\mathrm{Exp}(T) = \alpha_v(1) = \exp(T) \ ,$$

that is, it is the matrix exponential.

Ad representation

Recall that conjugation was defined as $C_g(x) = gxg^{-1}$. For a group element $g \in G$ and a matrix $T \in \mathcal{L}(G)$ It follows that

$$\mathrm{Ad}(g)(T) = T_e C_g(T) \stackrel{\text{Thm. (3.4)}}{=} \exp^{-1} \circ C_g \circ \exp(T) = \exp^{-1}(g e^T g^{-1}) = g \exp^{-1}(e^T) g^{-1} = gTg^{-1} \; .$$

So the simple conclusion is that, expressed in matrix language, the adjoint representation amounts to a conjugation

$$\mathrm{Ad}(g)(T) = gTg^{-1} \tag{3.27}$$

of the Lie algebra matrix T by the group elements g.

ad representation

From Theorem (3.2) and the fact that vector field commutators translate to matrix commutators, we know that for two matrices $T, S \in \mathcal{L}(G)$ we have

$$\mathrm{ad}(T)(S) = [T, S] \tag{3.28}$$

On the generators (T_i) we have $\mathrm{ad}(T_i)(T_j) = [T_i, T_j] = f_{ij}{}^k T_k$ so that the representing matrices for the generators, relative to the generator basis, are $[\mathrm{ad}(T_i)]^k{}_j = f_{ij}{}^k$, which is of course a version of Eq. (3.16).

Summary

In summary, for matrix Lie groups we have now translated a somewhat abstract story in differential geometry into a straightforward exercise in matrix computations. The generators T_i of the matrix Lie group G can be computed from Eq. (3.21) and they span the Lie algebra $\mathcal{L}(G)$ which is now a vector space of matrices with the matrix commutator as the commutator bracket. The exponential map is simply the matrix exponential and the adjoint acts by conjugation as in Eq. (3.27).

3.6 Simple examples: SU(2) and SO(3)

It is now time to practice some of the above ideas in the context of simple examples: the Lie groups SU(2) and SO(3) which, as we will see, are closely related.

Definition of groups

The group SU(2) consists of 2×2 unitary matrices with determinant one, so

$$\begin{aligned} \mathrm{SU}(2) &= \{U \in \mathrm{GL}(\mathbb{C}^2) \,|\, U^\dagger U = \mathbb{1}_2 \,, \;\; \det(U) = 1\} \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \,|\, \alpha, \beta \in \mathbb{C}, \; |\alpha|^2 + |\beta|^2 = 1 \right\} \cong S^3 \; . \end{aligned} \tag{3.29}$$

This is evidently a subgroup of $\mathrm{GL}(\mathbb{C}^2)$ since it contains the unit matrix, it is closed under multiplication and it contains the inverse. The explicit form in Eq. (3.29) can be obtained by inserting an arbitrary complex 2×2 matrix into the defining equations. It shows that, as a manifold, SU(2) is the same as the three-sphere S^3. The orthogonal group O(3) and the rotation group SO(3) in three dimensions are defined as

$$\mathrm{O}(3) = \{R \in \mathrm{GL}(\mathbb{R}^3) \,|\, R^T R = \mathbb{1}_3\} \,, \qquad \mathrm{SO}(3) = \{R \in \mathrm{O}(3) \,|\, \det(R) = 1\} \; . \tag{3.30}$$

Exercise 3.5. *Show that* $\mathrm{SU}(2)$ *is a subgroup of* $\mathrm{GL}(\mathbb{C}^2)$*. Derive the explicit form for* $\mathrm{SU}(2)$ *matrices in Eq. (3.29). Further, show that* $\mathrm{O}(3)$ *and* $\mathrm{SO}(3)$ *are subgroups of* $\mathrm{GL}(\mathbb{R}^3)$*.*

The orthogonality condition $R^T R = \mathbb{1}_3$ implies that $\det(R) \in \{\pm 1\}$, so, with the matrix $P = -\mathbb{1}_3$, also referred to as **parity**, we can write the orthogonal group as a disjoint union

$$\mathrm{O}(3) = \mathrm{SO}(3) \cup P(\mathrm{SO}(3)) \; . \tag{3.31}$$

of rotations and rotations times parity. Since the determinant cannot jump from $+1$ to -1 along a continuous path, these two components are path disconnected.

Lie algebra of $\mathrm{SU}(2)$

To compute the Lie algebra we can either write down a parametrisation of the group and use Eq. (3.21) or, if the group is defined in terms of constraints, work out the linearised version of these constraints.

Exercise 3.6. *Start with the parametrisation (3.29) of* $\mathrm{SU}(2)$ *and use Eq. (3.21) to work out the* $\mathrm{SU}(2)$ *generators and Lie algebra.*

We use the latter method and write $U = \mathbb{1}_2 + T + \mathcal{O}(T^2)$, insert this into the defining equations and read off the linear constraint on T.

$$\begin{aligned} U^\dagger U = \mathbb{1}_2 + T + T^\dagger + \mathrm{O}(T^2) &\stackrel{!}{=} \mathbb{1}_2 \qquad \Rightarrow \qquad T = -T^\dagger \\ \det(U) = 1 + \mathrm{tr}(T) + \mathcal{O}(T^2) &\stackrel{!}{=} 1 \qquad \Rightarrow \qquad \mathrm{tr}(T) = 0 \end{aligned}$$

(Of course, this is just a quick way of differentiating the defining relations with respect to the parameters t^i and evaluating the results at $t = 0$. For example, differentiating $U^\dagger U = \mathbb{1}_2$ in this way leads to $\partial_i U^\dagger U + U^\dagger \partial_i U = 0$ and evaluating at $t = 0$, remembering that $U(0) = \mathbb{1}_2$ and $\partial_i U(0) = T_i$ gives $T_i^\dagger + T_i = 0$.) Hence, the Lie algebra of $\mathrm{SU}(2)$ is given by the three-dimensional vector space of anti-hermitian [2] traceless 2×2 matrices

$$\mathrm{su}(2) = \mathcal{L}(\mathrm{SU}(2)) = \{T \in \mathrm{End}(\mathbb{C}^2) \,|\, T = -T^\dagger \; , \;\; \mathrm{tr}(T) = 0\} = \mathrm{Span}(\tau_1, \tau_2, \tau_3) \; , \tag{3.32}$$

where $\tau_i = -i\sigma_i/2$ and σ_i are the Pauli matrices (which are hermitian, traceless and linearly independent and can hence be used to form a basis of $\mathrm{su}(2)$).

This is probably a good place to collect some properties of Pauli matrices before we carry on. We have already seen the key relation

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_2 + i\epsilon_{ijk}\sigma_k \; . \tag{3.33}$$

Exchanging the indices i and j in this equation and subtracting from or adding to the original leads to the commutator and anti-commutator of the Pauli matrices.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \; , \qquad \{\sigma_i, \sigma_j\} = 2\mathbb{1}_2\delta_{ij} \; . \tag{3.34}$$

In particular, these relations imply that the Pauli matrices square to one and anti-commute, so

$$\sigma_i^2 = \mathbb{1}_2 \; , \qquad \sigma_i\sigma_j = -\sigma_j\sigma_i \quad \text{for} \quad i \neq j \tag{3.35}$$

[2] By writing $U = \mathbb{1}_2 + T + \cdot$ we are using the convention prevalent in mathematics. In the physics literature $U = \mathbb{1}_2 + iT + \cdot$ is more common and this leads to hermitian, rather than anti-hermitian matrices T.

both of which are useful properties for computation. Taking the trace of Eq. (3.33) (and remembering that σ_i have vanishing trace) leads to

$$\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}\ . \tag{3.36}$$

The complex conjugate of the Pauli matrices is $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$ and $\sigma_3^* = \sigma_3$ and, using Eq. (3.35), this can also be written more concisely as

$$\sigma_2\sigma_i^*\sigma_2 = -\sigma_i\ . \tag{3.37}$$

The Pauli matrix commutation relations (3.34) immediately translate into the su(2) commutation relations

$$[\tau_i,\tau_j] = \epsilon_{ij}{}^k\tau_k \tag{3.38}$$

and, hence, the su(2) structure constants are given by $f_{ij}{}^k = \epsilon_{ij}{}^k$. This means the representation matrices of the adjoint representation ad, relative to the basis (τ_i), are $[\mathrm{ad}(\tau_i)]^k{}_j = \epsilon_{ij}{}^k$ or, written as matrices $T_i = [\mathrm{ad}(\tau_i)]$, they are

$$T_1 = \begin{pmatrix} 0&0&0\\ 0&0&-1\\ 0&1&0\end{pmatrix}\ ,\quad T_2 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ -1&0&0\end{pmatrix}\ ,\quad T_3 = \begin{pmatrix} 0&-1&0\\ 1&0&0\\ 0&0&0\end{pmatrix}\ . \tag{3.39}$$

Lie algebra of (S)O(3)

We proceed as before and write $R = \mathbb{1}_3 + T + \mathcal{O}(T^2)$ and work out the linearised constraint

$$R^TR = \mathbb{1}_3 + T + T^T + \mathcal{O}(T^2) \stackrel{!}{=} \mathbb{1}_3 \quad\Rightarrow\quad T = -T^T\ .$$

Since the trace of antisymmetric matrices vanishes automatically the condition $\det(R) = 1$ does not lead to any additional constraint. We conclude that the Lie algebra of O(3) and SO(3) are the same (not surprising given they only differ in their global structure but are the same near the identity) and are given by the three-dimensional vector space of anti-symmetric matrices

$$(\mathrm{s})\mathrm{o}(3) = \mathcal{L}((\mathrm{S})\mathrm{O}(3)) = \{T\in\mathrm{End}(\mathbb{R}^3)\,|\,T=-T^T\} = \mathrm{Span}(T_1,T_2,T_3)\ , \tag{3.40}$$

with the matrices T_i in Eq. (3.39) as a basis. Their commutation relations are

$$[T_i,T_j] = \epsilon_{ij}{}^kT_k \tag{3.41}$$

so we have the same structure constants, $f_{ij}{}^k = \epsilon_{ij}{}^k$, as for su(2). This means that the linear map defined by $\tau_i \mapsto T_i$ is a Lie algebra (iso)morphism and that so(3) is, in fact, the adjoint representation of su(2).

Exponential map

For the case of SU(2) the exponential map can be explicitly worked out, thanks to the relation (3.33), and this results in

$$\exp(v^iT_i) = \cos\left(\frac{|v|}{2}\right)\mathbb{1}_2 - i\sin\left(\frac{|v|}{2}\right)\frac{v^i\sigma_i}{|v|}\ . \tag{3.42}$$

By comparing with Eq. (3.29) it is not hard to see that the image of exp is, in fact, all of SU(2).

Fundamental representation

For matrix groups, the representation defined by the matrices themselves is also called the **fundamental representation**. Hence, the fundamental representation R of SU(2) is two-dimensional. Its complex conjugate representation R^* satisfies

$$R^*(U) = U^* = \exp(T^*) \overset{(3.37)}{=} \sigma_2 \exp(T)\sigma_2^{-1} = \sigma_2 U \sigma_2^{-1} \; . \tag{3.43}$$

This shows that the fundamental representation R of SU(2) and its complex conjugate representation R^* are equivalent. However, R is not real, since the matrices U contain complex entries, but rather pseudo-real. The fundamental representation of SO(3) is three dimensional and this is clearly a real representation.

Adjoint representation

We have seen above, that the ad representation of the Lie algebra su(2) leads to the Lie algebra so(3). We would now like to study the corresponding relationship at the level of groups by constructing the Ad representation of SU(2). We recall that the adjoint is a representation of the group on its own Lie algebra so, for SU(2), it is a representation on the vector space $\mathrm{su}(2) = \mathrm{Span}(\tau_1, \tau_2, \tau_3)$. We identify this vector space with $\mathbb{R}^3$ by introducing the map

$$\varphi : \mathbb{R}^3 \xrightarrow{\cong} \mathrm{su}(2) \; , \qquad \varphi(X) = X^i \tau_i \; .$$

With this definition and thanks to Eq. (3.36) the dot product in $\mathbb{R}^3$ can be written as

$$X \cdot Y = -2\,\mathrm{tr}(\varphi(X)\varphi(Y)) \; . \tag{3.44}$$

Instead of looking at the adjoint representation $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{GL}(\mathrm{su}(2))$ with $\mathrm{Ad}(U)(T) = UTU^\dagger$ directly we consider the equivalent representation $R : \mathrm{SU}(2) \to \mathrm{GL}(\mathbb{R}^3)$ defined by

$$R(U) = \varphi^{-1} \circ \mathrm{Ad}(U) \circ \varphi \quad \Leftrightarrow \quad \sigma \cdot (R(U)X) = U(X \cdot \sigma)U^\dagger \; , \tag{3.45}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a formal vector which contains the three Pauli matrices. It is not too hard to verify that the representation R is unitary, relative to the dot product.

$$\begin{aligned}(R(U)X)\cdot(R(U)Y) &\overset{(3.44)}{=} -2\,\mathrm{tr}(\varphi(R(U)X)\varphi(R(U)Y)) \overset{(3.45)}{=} -2\,\mathrm{tr}(\mathrm{Ad}(U)(\varphi(X))\,\mathrm{Ad}(U)(\varphi(Y))) \\ &= -2\,\mathrm{tr}(U\varphi(X)U^\dagger U\varphi(Y)U^\dagger) = -2\,\mathrm{tr}(\varphi(X)\varphi(Y)) \overset{(3.44)}{=} X\cdot Y\end{aligned}$$

The immediate conclusion is that $R(U) \in \mathrm{O}(3)$, so the matrices $R(U)$ are orthogonal. In fact, since $R(\mathbb{1}_2) = \mathbb{1}_3$ and $\mathrm{SU}(2) \cong S^3$ is path connected it is clear that $R(U)$ must be in the component of O(3) path connected to the identity, that is, $R(U) \in \mathrm{SO}(3)$. The kernel of R can be worked out from Schur's Lemma. For $U \in \mathrm{Ker}(R)$ we have

$$R(U) = \mathbb{1}_3 \quad \Leftrightarrow \quad T = UTU^\dagger \;\; \forall T \in \mathrm{su}(2) \quad \Leftrightarrow \quad [T, U] = 0 \;\; \forall T \in \mathrm{su}(2) \quad \Leftrightarrow \quad U = \lambda \mathbb{1}_2$$

The only values of λ for which $\lambda\mathbb{1}_2 \in \mathrm{SU}(2)$ are $\lambda = \pm 1$, so we conclude that $\mathrm{Ker}(R) = \{\pm \mathbb{1}_2\} \cong \mathbb{Z}_2$. It can also be shown that $R : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective, so the isomorphism theorem for groups implies that

$$\mathrm{SO}(3) \cong \frac{\mathrm{SU}(2)}{\mathbb{Z}_2} \; . \tag{3.46}$$

So we see that SO(3) is indeed the adjoint representation of SU(2), as suggested by the results on the Lie algebra level. Eq. (3.46) is sometimes expressed by saying that SU(2) is a two-fold cover of SO(3): Any pair $U, -U \in \mathrm{SU}(2)$ is mapped to the same rotation matrix $R(U) = R(-U) \in \mathrm{SO}(3)$.

Chapter 4

Lie algebras

It is now time to look at Lie algebras in their own right and collect some information about their structure. In this chapter we will be working with a Lie algebra $\mathcal{L}$ with bracket $[\cdot, \cdot]$, typical elements denoted by $T, S, \ldots$, basis (T_i), where $i, j, \ldots = 1, \ldots, n$, and commutators $[T_i, T_j] = f_{ij}{}^k T_k$. The adjoint representation $\mathrm{ad} : \mathcal{L} \to \mathrm{GL}(\mathcal{L})$ is given by $\mathrm{ad}(S)(T) = [S, T]$.

4.1 Structure of Lie algebras

We need to start with a definition which collects the various notions we require to describe the structure of a Lie algebra.

Basic definitions and terminology

Definition 4.1. *For a Lie algebra $\mathcal{L}$ we have the following definitions:*

(i) *A subset $\mathcal{A} \subset \mathcal{L}$ is called a* ***(Lie) sub-algebra*** *iff $\mathcal{A}$ is a linear subspace and $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$.*

(ii) *A sub-algebra is called* ***Abelian*** *iff $[\mathcal{A}, \mathcal{A}] = 0$.*

(iii) *A sub-algebra is an* ***ideal*** *iff $[\mathcal{L}, \mathcal{A}] \subset \mathcal{A}$. An ideal is called non-trivial iff $\mathcal{A} \neq \{0\}, \mathcal{L}$.*

(iv) *The* ***derived series*** *$\{\mathcal{D}^k\mathcal{L}\}$ of $\mathcal{L}$ is defined by $\mathcal{D}^1\mathcal{L} = [\mathcal{L}, \mathcal{L}]$ and $\mathcal{D}^k\mathcal{L} = [\mathcal{D}^{k-1}\mathcal{L}, \mathcal{D}^{k-1}\mathcal{L}]$.*

(v) *$\mathcal{L}$ is called* ***solvable*** *iff $\mathcal{D}^k\mathcal{L} = 0$ for some k.*

The following definition sets up the most important structural features of Lie algebra, simplicity and semi-simplicity.

Definition 4.2. *A Lie algebra $\mathcal{L}$ is called*

(i) ***simple*** *iff it contains no non-trivial ideals.*

(ii) ***semi-simple*** *iff it contains no non-zero solvable ideals.*

An alternative and equivalent way to define semi-simplicity is:

Lemma 4.1. *$\mathcal{L}$ semi-simple $\iff$ $\mathcal{L}$ has no non-zero Abelian ideals.*

Proof. "$\Rightarrow$" We prove this indirectly, so assume that $\mathcal{L}$ has a non-zero Abelian ideal $\mathcal{A}$. Then $\mathcal{D}^1\mathcal{A} = [\mathcal{A}, \mathcal{A}] = 0$ implies that $\mathcal{A}$ is solvable, and therefore $\mathcal{L}$ is not semi-simple.

"$\Leftarrow$" It is straightforward to show (from the Jacobi identity) that for an ideal $\mathcal{A} \subset \mathcal{L}$ the entire derived series $\mathcal{D}^k\mathcal{A}$ consists of ideals. Now assume that $\mathcal{L}$ is not semi-simple and, hence, has a non-zero solvable ideal $\mathcal{A}$. Then there is a k such that $\mathcal{D}^{k-1}\mathcal{A} \neq 0$ and $\mathcal{D}^k\mathcal{A} = 0$ and it follows that $\mathcal{D}^{k-1}\mathcal{A}$ is a non-zero Abelian ideal. $\square$

Example 4.1: *(Lie algebras of $U(1)$ and $\mathrm{SU}(2)$)*

The Lie algebra of $U(1)$ is $u(1) = \mathbb{R}$ which is Abelian. Hence, $u(1)$ has a non-zero Abelian ideal (all of $u(1)$) and is not semi-simple.

The Lie algebra su(2) is spanned by matrices τ_i with $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$. If the write $a \cdot \tau = a_i\tau_i$ for a vector $a \in \mathbb{R}^3$, then $[a \cdot \tau, b \cdot \tau] = (a \times b) \cdot \tau$, where $\times$ denotes the cross product. This relation shows that neither a one-dimensional subspace $\mathrm{Span}(a \cdot \tau)$, nor a two-dimensional subspace $\mathrm{Span}(a \cdot \tau, b \cdot \tau)$ can be an ideal. Hence, su(2) is simple. □

Decomposition of Lie algebras

The next statement is a technical one which will allow us to define the radical of Lie algebra.

Lemma 4.2. *If $\mathcal{B}, \mathcal{C} \subset \mathcal{L}$ are solvable ideals, then so is $\mathcal{A} = \mathcal{B} + \mathcal{C}$.*

Proof. Let $\mathcal{B}, \mathcal{C}$ both be solvable ideals. Then $\mathcal{A} = \mathcal{B} + \mathcal{C}$ clearly is an ideal and all we need to show is that it is solvable. We start by observing that $\mathcal{D}^i\mathcal{A} = \mathcal{D}^1\mathcal{B} + \mathcal{D}^i\mathcal{C} + \mathcal{A}_i$, where $\mathcal{A}_i \subset \mathcal{B} \cap \mathcal{C}$. To see this, consider, for example, $\mathcal{D}^1\mathcal{A} = [\mathcal{B} + \mathcal{C}, \mathcal{B} + \mathcal{C}] = \mathcal{D}^1\mathcal{B} + \mathcal{D}^1\mathcal{C} + [\mathcal{B}, \mathcal{C}]$ and observe that $\mathcal{A}_1 := [\mathcal{B}, \mathcal{C}] \subset \mathcal{B} \cap \mathcal{C}$ since both $\mathcal{B}$ and $\mathcal{C}$ are ideals. For higher entries $\mathcal{D}^i\mathcal{A}$ in the derived series the statement follows analogously.

Since $\mathcal{B}$ and $\mathcal{C}$ are both solvable there exists a k with $\mathcal{D}^k\mathcal{B} = \mathcal{D}^k\mathcal{C} = 0$. It follows that $\mathcal{D}^k\mathcal{A} = \mathcal{A}_k \subset \mathcal{B} \cap \mathcal{C} \subset \mathcal{B}$. Hence, $\mathcal{D}^{2k}\mathcal{A} \subset \mathcal{D}^k\mathcal{B} = 0$ and therefore $\mathcal{A}$ is solvable. □

We now define the radical which is the obstruction to $\mathcal{L}$ being semi-simple.

Definition 4.3. *The sum of all solvable ideals in $\mathcal{L}$ is called the* ***radical*** *$\mathrm{rad}(\mathcal{L})$ of $\mathcal{L}$.*

It turns out, every Lie algebra can be written as a semi-direct sum of the radical and a semi-simple part.

Theorem 4.1. *(Levi) A Lie algebra $\mathcal{L}$ can be written as $\mathcal{L} = \mathcal{P} \oplus_S \mathcal{A}$, where $\mathcal{P} = \mathrm{rad}(\mathcal{L})$ is the radical and $\mathcal{A}$ is semi-simple. (The semi-direct product refers to a direct sum of vector spaces and commutators of the form $[\mathcal{P}, \mathcal{P}] \subset \mathcal{P}$, $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$ and $[\mathcal{P}, \mathcal{A}] \subset \mathcal{P}$.)*

Proof. Ref. [1], p. 499. □

Theorem 4.2. *A semi-simple Lie algebra $\mathcal{L}$ is a direct sum of $\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$ of simple Lie algebras $\mathcal{L}_i$. (The direct sum refers to a direct vector space sum and the commutators $[\mathcal{L}_i, \mathcal{L}_j] = 0$ for $i \neq j$.)*

Proof. Ref. [1], p. 480. □

Combining the previous two theorems we learn that every Lie algebra $\mathcal{L}$ can be written in the form

$$\mathcal{L} = \mathrm{rad}(\mathcal{L}) \oplus_S (\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k) \tag{4.1}$$

where the $\mathcal{L}_i$ are simple and commute with each other.

Example 4.2: *(Structure of Lie algebras)*

Consider the direct product group $G = G_1 \times G_2$ of two Lie groups G_1 and G_2. This is again a Lie group with Lie algebra $\mathcal{L}(G) \cong T_eG = T_eG_1 \oplus T_eG_2 \cong \mathcal{L}(G_1) \oplus \mathcal{L}(G_2)$ (and the sum is direct since vectors fields on G_1 commute with those on G_2). Hence, a direct product of Lie groups leads to a direct sum of the associated Lie algebras.

For example, the Lie algebra for SU(2) × SU(2)

$$\mathcal{L}(\mathrm{SU}(2)\times\mathrm{SU}(2)) = \underbrace{\overbrace{\mathrm{su}(2)}^{\text{simple}} \oplus \overbrace{\mathrm{su}(2)}^{\text{simple}}}_{\text{semi}-\text{simple}}$$

is a direct sum of two simple algebras (since su(2) is simple), so it is semi-simple. On the other hand, this Lie algebra for $U(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ is

$$\mathcal{L}(U(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)) = \mathcal{L} = \underbrace{u(1)}_{\text{radical}} \oplus \underbrace{\overbrace{\mathrm{su}(2)}^{\text{simple}} \oplus \overbrace{\mathrm{su}(2)}^{\text{simple}}}_{\text{semi}-\text{simple}}$$

is not semi-simple, since u(1) is a non-zero Abelian ideal. In fact, comparing with Eq. (4.1), $\mathrm{rad}(\mathcal{L}) = \mathrm{u}(1)$ is the radical and $\mathcal{L}_1 = \mathcal{L}_2 = \mathrm{su}(2)$ are the simple parts. □

The following statement relates simplicity and semi-simplicity to properties of the adjoint representation.

Proposition 4.1. *The adjoint representation* $\mathrm{ad}:\mathcal{L}\to\mathrm{End}(\mathcal{L})$ *satisfies:*

(i) *If* $\mathcal{L}$ *is semi-simple then* ad *is faithful.*

(ii) *If* $\mathcal{L}$ *is simple ad is irreducible.*

Proof. (i) We prove this indirectly and assume that ad is not faithful. This means that $\mathrm{Ker}(\mathrm{ad}) \neq 0$ and there must be a non-zero $T\in\mathcal{L}$ such that $\mathrm{ad}(T)(S) = [T,S] = 0$ for all $S\in\mathcal{L}$. This implies that $\mathrm{Span}(T)$ is a non-zero Abelian ideal so that $\mathcal{L}$ is not semi-simple.

(ii) Again, we do this indirectly and assume that ad is reducible. This means there exists a non-trivial subspace $\mathcal{A}\subset\mathcal{L}$ such that $\mathrm{ad}(T)\mathcal{A}\subset\mathcal{A}$ for all $T\in\mathcal{L}$. It follows that $[\mathcal{L},\mathcal{A}]\subset\mathcal{A}$ so $\mathcal{A}$ is a non-trivial ideal and $\mathcal{L}$ is not simple. □

4.2 The Killing form

A Lie algebra carries a symmetric bi-linear form, the **Killing form**, which plays an important role in analysing the structure of Lie algebras and is defined as follows.

Definition of Killing form

Definition 4.4. *The symmetric bilinear form* $\Gamma:\mathcal{L}\times\mathcal{L}\to\mathbb{C}$ *defined by*

$$\Gamma(T,S) = \mathrm{tr}(\mathrm{ad}(T)\mathrm{ad}(S)) \tag{4.2}$$

is called the ***Killing form*** *of* $\mathcal{L}$

Remark 4.1. *We can use the representation matrices (3.17) of the adjoint representation to compute the Killing form* $\gamma_{ij} = \Gamma(T_i,T_j)$ *relative to a basis* (T_i)*.*

$$\gamma_{ij} = \Gamma(T_i,T_j) = \mathrm{tr}(\mathrm{ad}(T_i)\mathrm{ad}(T_j)) = [\mathrm{ad}(T_i)]^l_k\,[\mathrm{ad}(T_j)]^k_l = f_{ik}{}^l f_{jl}{}^k \tag{4.3}$$

Then, the Killing form of two Lie algebra elements $T = v^iT_i$ *and* $S = w^jT_j$ *can be written as* $\Gamma(T,S) = \gamma_{ij}v^iw^j$.

It turns out the adjoint representation is anti self-adjoint relative to the Killing form.

Proposition 4.2. *(Property of Killing form) The Killing form satisfies*

$$\Gamma(T, \mathrm{ad}(U)S) = -\Gamma(\mathrm{ad}(U)T, S) \tag{4.4}$$

Proof.

$$\begin{aligned}\Gamma(T, \mathrm{ad}(U)S) &= \Gamma(T, [U, S]) = \mathrm{tr}(\mathrm{ad}(T) \circ \mathrm{ad}([U, S])) = \mathrm{tr}(\mathrm{ad}(T) \circ [\mathrm{ad}(U), \mathrm{ad}(S)]) \\ &= \mathrm{tr}(\mathrm{ad}(T) \circ \mathrm{ad}(U) \circ \mathrm{ad}(S)) - \mathrm{tr}(\mathrm{ad}(T) \circ \mathrm{ad}(S) \circ \mathrm{ad}(U)) \\ &= \mathrm{tr}([\mathrm{ad}(T), \mathrm{ad}(U)] \circ \mathrm{ad}(S)] = \mathrm{tr}(\mathrm{ad}([T, U] \circ \mathrm{ad}(S)) = \Gamma([T, U], S) \\ &= -\Gamma(\mathrm{ad}(U)T, S)\end{aligned}$$

□

Kiling form and Lie algebra structure

The Killing form Γ is called (non-degenerate) iff $\Gamma(T, S) = 0$ for all $T \in \mathcal{L}$ implies that $S = 0$. Equivalently, Γ is non-degenerate iff the matrix (γ_{ij}) is invertible. The Killing form can be used to decide whether a Lie algebra is semi-simple.

Theorem 4.3. *A Lie algebra $\mathcal{L}$ is semi-simple iff Γ is non-degenerate.*

Proof. "$\Leftarrow$" Assume Γ is non-degenerate and $\mathcal{A}$ is an Abelian ideal. We want to show that $\mathcal{A}$ is the zero ideal. To do this set up a basis (T_a, T_α), where (T_a) is a basis of $\mathcal{A}$ and (T_α) a basis of the remainder. For $T \in \mathcal{L}$ and $S \in \mathcal{A}$ we have

$$\mathrm{ad}(T) \circ \mathrm{ad}(S)(T_a) = [T, [S, T_a]] = [T, 0] = 0\,, \qquad \mathrm{ad}(T) \circ \mathrm{ad}(S)(T_\alpha) = [T, [S, T_\alpha]] \in \mathcal{A}$$

and, hence, the Killing form is given by

$$\Gamma(T, S) = \mathrm{tr}(\mathrm{ad}(T)\mathrm{ad}(S)) = \mathrm{tr}\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} = 0$$

for all $T \in \mathcal{L}$. Since Γ is non-degenerate this implies that $S = 0$ and, hence, that $\mathcal{A} = \{0\}$. For the other direction (which can be shown along similar lines) see Ref. [1], p. 480. □

It is useful to have some final statement about the sign of the quadratic form associated to the Killing form.

Theorem 4.4. *If G is compact, then the Killing form Γ on $\mathcal{L}(G)$ is negative semi-definite (that is, $\Gamma(T, T) \leq 0$ for all $T \in G$).*

Proof. See Ref. [2], p. 214. □

4.3 Some useful properties of structure constants

Practical applications are often formulated in terms of a basis (T_i) of the Lie algebra $\mathcal{L}$, where $i = 1, \ldots, n$, structure constants ${f_{ij}}^k$ and the Killing form $\gamma_{ij} = {f_{ik}}^l {f_{jl}}^k$. If $\mathcal{L}$ is semi-simple then γ_{ij} is invertible and we can also introduce its inverse γ^{ij}. It is useful to translate some of the previous results into this language.

Jacobi identity for structure constants

The Jacobi identity $[[T_i,T_j],T_k]+[[T_j,T_k],T_i]+[[T_k,T_i],T_j]=0$ together with the commutation relations $[T_i,T_j]=f_{ij}{}^kT_k$ translates into the relation

$$f_{ij}{}^l f_{kl}{}^n + f_{jk}{}^l f_{il}{}^n + f_{ki}{}^l f_{jl}{}^n = 0 \tag{4.5}$$

for the structure constants. Of course the structure constants are anti-symmetric in the first two indices, so

$$f_{ij}{}^k = -f_{ji}{}^k\ . \tag{4.6}$$

Totally anti-symmetric structure constants

To get to a stronger statement, we introduce the structure constants

$$f_{ijk} = \gamma_{kl} f_{ij}{}^l \tag{4.7}$$

with three lower indices, where the Killing metric is used to lower the last index. A short calculation shows that

$$\begin{aligned} f_{ijk} &\overset{(4.7)}{=} \gamma_{kl} f_{ij}{}^l \overset{(4.3)}{=} f_{km}{}^n f_{ij}{}^l f_{ln}{}^m \overset{(4.5)}{=} f_{km}{}^n (f_{jn}{}^l f_{il}{}^m + f_{ni}{}^l f_{jl}{}^m) \\ &= f_{km}{}^n f_{jn}{}^l f_{il}{}^m + f_{mk}{}^n f_{ni}{}^l f_{lj}{}^m = \mathrm{tr}(f_k f_j f_i) + \mathrm{tr}(\tilde f_k \tilde f_i \tilde f_j)\ , \end{aligned}$$

where f_i ($\tilde f_i$) is the matrix with entries $f_{ij}{}^k$ ($f_{ji}{}^k$). Cyclicity of the trace shows that the structure constants f_{ijk} are unchanged under cyclic permutations of the three indices. Together with anti-symmetry in the first two indices this implies that

$$f_{ijk} \quad \text{is totally anti-symmetric}\ . \tag{4.8}$$

Quadratic Casimir

If $\mathcal{L}$ is semi-simple we can define the **quadratic Casimir operator**

$$C = \gamma^{ij} T_i T_j \tag{4.9}$$

Its relevance is that it commutes with the entire Lie algebra.

Theorem 4.5. *The Casimir operator* (4.9) *satisfies* $[C,T]=0$ *for all* $T\in\mathcal{L}$,

Proof.

$$\begin{aligned} [C,T_l] &= \gamma^{ij}[T_iT_j,T_l] = \gamma^{ij}T_i[T_j,T_l] + \gamma^{ij}[T_i,T_l]T_j = \gamma^{ij} f_{jl}{}^m T_i T_m + \gamma^{ij} f_{il}{}^m T_m T_j \\ &= \gamma^{ij} f_{jl}{}^m (T_iT_m + T_mT_i) \overset{(4.7)}{=} \gamma^{ij}\gamma^{mn}(T_iT_m+T_mT_i) f_{jln} \overset{(4.8)}{=} 0 \end{aligned}$$

The last equality follows from the fact that f_{jln} is antisymmetric in (j,n), while the remainder of the expression is symmetric. □

Corollary 4.1. *If the Lie algebra is irreducible then* $C=\lambda\mathbb{1}$.

Proof. This is a direct consequence of Theorem 4.5 and Schur's Lemma applied at the level of the algebra. □

The number λ - the value of the Casimir operator - is a characteristic number for the Lie algebra under consideration.

Physics conventions

Physics applications often rely on a specific convention for the generators and structure constants but the underlying assumptions are rarely spelled out explicitly. The purpose of this section is to do so. We start by assuming that the Lie group G is compact and that its algebra $\mathcal{L}(G)$ is simple. Then, combining Theorems 4.3 and 4.4, we learn that this Killing form is non-degenerate and negative semi-definite and, hence, non-degenerate and negative definite. This means we can choose a basis (T_i) of $\mathcal{L}(G)$ such that

$$\gamma_{ij} = -\delta_{ij} \qquad \Rightarrow \qquad f_{ij}{}^{k} = -f_{ijk}\ . \tag{4.10}$$

As a result the structure constants $f_{ij}{}^{k}$ which appear in the commutation relations are completely anti-symmetric, in the same way as their lower index counterparts f_{ijk}.

Consider an irrep $r : \mathcal{L}(G) \to \mathrm{End}(\mathbb{F}^d)$, where $d = \dim(r)$ and write the representation matrices of the basis as $T_i^{(r)} = r(T_i)$. For the Casimir $C^{(r)}$ in the representation r we have

$$C^{(r)} \overset{(4.9)}{=} -\sum_i (T_i^{(r)})^2 \overset{\text{Cor. 4.1}}{=} C(r)\mathbb{1}_d\ . \tag{4.11}$$

where the value $C(r)$ of the Casimir characterises the representation r. It turns out that, given these conventions, the representation matrices $T_i^{(r)}$ satisfy a nice normalisation condition with respect to the trace.

Proposition 4.3. *With the above conventions we have*

$$\mathrm{tr}\left(T_i^{(r)}T_j^{(r)}\right) = -c(r)\,\delta_{ij} \tag{4.12}$$

where $c(r)$ is a number, also called the ***index*** *of the representation r.*

Proof. We define the normalisation matrix $M_{jk} = \mathrm{tr}(T_j^{(r)}T_k^{(r)})$ and want to show, using Schur's Lemma, that it is proportional to the unit matrix.

$$\begin{aligned}([T_i^{(ad)}, M])_{jk} &= (T_i^{(ad)})_{jl}\mathrm{tr}(T_l^{(r)}T_k^{(r)}) - tr(T_j^{(r)}T_l^{(r)})(T_i^{(ad)})_{lk} = \mathrm{tr}(f_{ijl}T_l^{(r)}T_k^{(r)}) - \mathrm{tr}(f_{ilk}T_j^{(r)}T_l^{(r)})\\ &= \mathrm{tr}([T_i^{(r)}, T_j^{(r)}]T_k^{(r)} + T_j^{(r)}[T_i^{(r)}, T_k^{(r)}]) = \mathrm{tr}([T_i^{(r)}, T_j^{(r)}T_k^{(r)}]) = 0\end{aligned}$$

Schur's Lemma then implies that $M = \lambda\mathbb{1}_d$. □

We have now found two numbers, the Casimir $C(r)$ and the index $c(r)$ which are associated to an irrep r. How are they related? The short calculation

$$\dim(r)C(r) = \mathrm{tr}(C^{(r)}) = -\sum_i \mathrm{tr}\left((T_i^{(r)})^2\right) = \sum_i c(r) = \dim(\mathrm{ad})\,c(r)$$

shows that

$$c(r) = \frac{\dim(r)}{\dim(\mathrm{ad})}C(r)\ . \tag{4.13}$$

4.4 Example SU(2) - again

Killing form, Casimir and index

We can now add a few pieces to the discussion of SU(2) and its algebra su(2). Recall that the latter consists of the 2×2 anti-hermitian, traceless matrices and is spanned by $\tau_i = -i\sigma_i/2$, where $i = 1, 2, 3$. The structure constants are $f_{ij}{}^k = \epsilon_{ij}{}^k$ and, hence, we have for the Killing form

$$\gamma_{ij} = f_{ik}{}^l f_{jl}{}^k = \epsilon_{ik}{}^l \epsilon_{jl}{}^k = -2\delta_{ij} \ . \tag{4.14}$$

(Apart from the factor 2 this realises the convention (4.10).) Since $\sigma_i^2 = \mathbb{1}_2$ this means the Casimir in the fundamental representation is

$$C = \gamma^{ij}\tau_i\tau_j = \frac{3}{8}\mathbb{1}_2 \ , \tag{4.15}$$

so that $C(\text{fund}) = 3/8$ and, from Eq. (4.13) (with $\dim(\text{fund}) = 2$ and $\dim(\text{ad}) = 3$), the index is $c(\text{fund}) = 1/4$. We note that γ_{ij} is invertible so, from Theorem 4.3, we conclude that su(2) is semi-simple (in fact, earlier arguments have already shown it is simple).

A new basis for su(2)

Denote by $T_i = r(\tau_i)$ the matrices in any su(2) representation, so that $[T_i, T_j] = \epsilon_{ij}{}^k T_k$. Let us introduce a new basis $(H, E_\pm)$ on the Lie algebra defined by

$$E_\pm = \frac{1}{2}(T_1 \pm iT_2) \ , \qquad H = iT_3 \ . \tag{4.16}$$

In the context of quantum mechanics, these are also called $J_i = -iT_i$ and $J_\pm = J_1 \pm iJ_2$ while the Casimir is $C = \gamma^{ij}T_iT_j = -J_iJ_i/2 = -J^2/2$. The definitions (4.16) are not quite as innocent as they look. Our original Lie algebra su(2) is a vector space over $\mathbb{R}$, so, with su(2) we are not really entitled to redefinitions which include factors of i. Instead, Eqs. (4.16) should be viewed as equations in $\text{su}(2)_\mathbb{C} = \text{su}(2) + i\,\text{su}(2)$, the **complexification** of su(2) and $(H, E_\pm)$ should be viewed as a basis on this complexification. Note that $\text{su}(2)_\mathbb{C}$ consists of the sum of all anti-hermitian and hermitian traceless matrices, so

$$\text{su}(2)_\mathbb{C} = \{T \in \text{End}(\mathbb{C}^2) \,|\, \text{tr}(T) = 0\} \ . \tag{4.17}$$

The commutation relations of our new basis elements are

$$[H, H] = 0 \ , \quad [H, E_\pm] = \pm E_\pm \ , \quad [E_+, E_-] = \frac{1}{2}H \ . \tag{4.18}$$

Representations of $\text{su}(2)_\mathbb{C}$

Finding the irreps of $\text{su}(2)_\mathbb{C}$ is a standard topic dealt with in quantum mechanics (for a nice account see, for example, the appendix of Ref. [10], vol. 2). Here, we simply recap the main results.

The irreps R_j of $\text{su}(2)_\mathbb{C}$ are labelled by an integer or half-integer $j \in \mathbb{Z}/2$ and are characterised by the value $J^2 = j(j+1)$ of the Casimir. The vector space V_j on which R_j acts is spanned by vectors $|jm\rangle$, where $j = -m, -m+1, \ldots, m-1, m$, with $\langle jm|jm'\rangle = \delta_{mm'}$. Hence, the dimension of the representation R_j is $\dim(R_j) = 2j + 1$. More specifically, the representation matrices $(J_3, J_\pm)$ act on this basis as

$$J_3|jm\rangle = m\,|jm\rangle\ , \qquad J_\pm|jm\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|jm \pm 1\rangle\ . \tag{4.19}$$

The basis idea for how to arrive at these results is evident from the last two equations. The basis $|jm\rangle$ is obtained as a basis of eigenvectors of J_3, with eigenvalues m, while $J_\pm$ act as "raising" and "lowering" operators, changing the eigenvalue by ± 1. The relations (4.19) can be used to compute explicit representation matrices for every R_j.

The Clebsch-Gordan decomposition for these representations is also well-known (in quantum mechanics referred to as "coupling of angular momenta") and is given by

$$R_{j_1} \otimes R_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} R_j\ . \tag{4.20}$$

In particular, this means that we have two preferred bases on $R_{j_1} \otimes R_{j_2}$, the obvious basis of tensor states $|j_1 m_1 j_2 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$ and the basis $|jm\rangle$ adapted to the right-hand side of Eq. (4.20). The relation between those bases

$$|jm\rangle = \sum_{m_1,m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|jm\rangle\ , \tag{4.21}$$

is sometimes also referred to as Clebsch-Gordan decomposition and the coefficients $\langle j_1 j_2 m_1 m_2|jm\rangle$ are called **Clebsch-Gordan coefficients**. They can be computed by standard methods familiar from quantum mechanics - basically, by applying J_- successively and using Eqs. (4.19).

4.5 The Cartan-Weyl basis

The method for constructing the irreps of $\mathrm{su}(2)_\mathbb{C}$ involves studying eigenvectors and eigenvalues of H and introducing raising and lowering operators $E_\pm$ which map between those eigenvectors. We would now like to generalise this method to an an arbitrary semi-simple, complex Lie algebra $\mathcal{L}$. The first step is to introduce the generalisation of H in the $\mathrm{su}(2)_\mathbb{C}$ case. This now becomes a subalgebra of $\mathcal{L}$ and, since the aim is to simultaneously diagonalise the elements in this subalgebra, it is required to be Abelian.

Cartan subalgebra, roots and Cartan decomposition

Definition 4.5. *A maximal, diagonalisable Abelian subalgebra $\mathcal{H} \subset \mathcal{L}$ is called a* ***Cartan subalgebra****. The dimension of $\mathcal{H}$ is called the* ***rank*** *of $\mathcal{L}$, denoted by $\mathrm{rk}(\mathcal{L}) = \dim(\mathcal{H})$.*

Of course there are a number of things to clarify about this definition. W need to worry about the existence and construction of the Cartan subalgebra and, since it turns out it is not unique, about whether $\dim(\mathcal{H})$ is well-defined. App. D of Ref. [1] shows that $\mathcal{H}$ always exists and that the rank, $\mathrm{rk}(\mathcal{L}) = \dim(\mathcal{H})$, is indeed well-defined. The requirement that $\mathcal{H}$ be diagonalisable means that the adjoint actions of all $H \in \mathcal{H}$ can be diagonalised simultaneously. This means we can study the simultaneous eigenvectors $T \in \mathcal{L}$ which satisfy the equation

$$\mathrm{ad}(H)(T) = \alpha(H)T \quad \text{for all} \quad H \in \mathcal{H}\ . \tag{4.22}$$

The eigenvalue $\alpha(H)$ depends on H linearly and hence $\alpha \in \mathcal{H}'$ is a linear functional on $\mathcal{H}$.

Definition 4.6. *(Roots) A non-zero linear functional $\alpha \in \mathcal{H}'$ is called a* ***root*** *of the Lie algebra $\mathcal{L}$ if there is a non-zero $T \in \mathcal{L}$ such that Eq. (4.22) is satisfied. The set $\Delta = \{\alpha \in \mathcal{H}' \,|\, \alpha \text{ is a root}\}$ is called the* ***root space*** *of $\mathcal{L}$. The lattice generated by Δ (that, is all integer linear combinations of elements in Δ) is called the* ***root lattice****, Λ_R.*

If we denote by $\mathcal{L}_\alpha \subset \mathcal{L}$ the eigenspace for root α then the Lie algebra can be written as

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Delta} \mathcal{L}_\alpha \,, \tag{4.23}$$

and this is called the **Cartan decomposition** of $\mathcal{L}$. The Cartan subalgebra is sometimes written as $\mathcal{L}_0 = \mathcal{H}$ (since its elements corresponds to eigenvectors of $\mathrm{ad}(H)$ with eigenvalues 0). Note, however, that, by definition, 0 is not a root.

Structure of Cartan decomposition

The Cartan decomposition is consistent with the commutator. To see this start with $T \in \mathcal{L}_\alpha$ and $S \in \mathcal{L}_\beta$ so that $[H,T] = \alpha(H)T$ and $[H,S] = \beta(H)S$. It follows that

$$[H,[T,S]] \overset{\text{Jacobi}}{=} [T,[H,S]] - [S[H,T]] \overset{(4.22)}{=} \beta(H)[T,S] - \alpha(H)[S,T] = (\alpha(H)+\beta(H))\,[T,S]$$

and hence

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subset \mathcal{L}_{\alpha+\beta} \,. \tag{4.24}$$

This result has direct implications for the relation between the Cartan decomposition and the Killing form. Consider eigenvectors $T \in \mathcal{L}_\alpha$, $S \in \mathcal{L}_\beta$ and $U \in \mathcal{L}_\gamma$ so that

$$\mathrm{ad}(T) \circ \mathrm{ad}(S)(U) = [T,[S,U]] \in \mathcal{L}_{\alpha+\beta+\gamma} \,.$$

If $\alpha + \beta \neq 0$ then this result means that $\mathrm{ad}(T) \circ \mathrm{ad}(S)$ has vanishing diagonal elements so that $\Gamma(T,S) = \mathrm{tr}(\mathrm{ad}(T) \circ \mathrm{ad}(S)) = 0$. We conclude that

$$\mathcal{L}_\alpha \perp \mathcal{L}_\beta \quad \text{for} \quad \alpha + \beta \neq 0 \,, \tag{4.25}$$

that is any two eigenspaces are orthogonal, relative to the Killing form, as long as their roots do not sum to zero. The Cartan decomposition leads to a number of further important properties which are summarised in the following theorem.

Theorem 4.6. *(Structure of the Cartan decomposition)*

- *(i)* $\Gamma|_{\mathcal{H}\times\mathcal{H}}$ *is non-degenerate.*
  For all $\alpha \in \mathcal{H}'$ there exists a unique $H_\alpha \in \mathcal{H}$ with $\Gamma(H, H_\alpha) = \alpha(H)$ for all $H \in \mathcal{H}$.
  We define the inner product of roots as $(\alpha, \beta) = \Gamma(H_\alpha, H_\beta)$.
- *(ii)* *If Δ contains α, it also contains $-\alpha$.*
- *(iii)* *For $T \in \mathcal{L}_\alpha$ and $S \in \mathcal{L}_{-\alpha}$, we have $[T,S] = \Gamma(T,S)H_\alpha$.*
  One can choose T, S such that $\Gamma(T,S) = 1$.
- *(iv)* $\dim(\mathcal{L}_\alpha) = 1$ *for all $\alpha \in \Delta$.*
- *(v)* *Let $\alpha \in \Delta$. Then, from $\{k\alpha \,|\, k \in \mathbb{Z}\}$, only α and $-\alpha$ are roots.*
- *(vi)* *For $H, \tilde{H} \in \mathcal{H}$, we have $\Gamma(H, \tilde{H}) = \sum_{\alpha \in \Delta} \alpha(H)\alpha(\tilde{H})$.*
- *(vii)* *Δ contains a basis of $\mathcal{H}'$ ("roots span root space")*

Proof. (i) Since $\mathcal{L}$ is semi-simple we know that the Killing form is non-degenerate on $\mathcal{L}\times\mathcal{L}$. To show that the Killing form is non-degenerate on $\mathcal{H}\times\mathcal{H}$ we start by assuming that $\Gamma(H,\tilde{H})=0$ for $H\in\mathcal{H}$ and all $\tilde{H}\in\mathcal{H}$. We need to show this implies that $H=0$. From the Cartan decomposition (4.23), write any $S\in\mathcal{L}$ as $S=\tilde{H}+\sum_{\alpha\in\Delta}S_\alpha$, where $\tilde{H}\in\mathcal{H}$ and $S_\alpha\in\mathcal{L}_\alpha$. It follows from Eq. (4.25) that

$$\Gamma(H,S)=\Gamma(H,\tilde{H})+\sum_{\alpha\in\Delta}\Gamma(H,S_\alpha)\overset{(4.25)}{=}0$$

and, hence, non-degeneracy of Γ on $\mathcal{L}\times\mathcal{L}$ implies that $H=0$. Hence, $\Gamma|_{\mathcal{H}\times\mathcal{H}}$ is non-degenerate and, following standard linear algebra, it defines an isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ as indicated.

(ii) Assume that $-\alpha\notin\Delta$. Then Eq. (4.25) implies that $\mathcal{L}_\alpha\perp\mathcal{L}_\beta$ for all $\beta\in\Delta$ as well as $\mathcal{L}_\alpha\perp\mathcal{H}$ so that $\mathcal{L}_\alpha\perp\mathcal{L}$. However, this is a contradiction, since Γ is non-degenerate.

(iii) Let $H\in\mathcal{H}$, $T\in\mathcal{L}_\alpha$ and $S\in\mathcal{L}_{-\alpha}$ and work out

$$\Gamma(H,[T,S])=\Gamma([H,T],S)=\alpha(H)\Gamma(T,S)=\Gamma(H,H_\alpha)\Gamma(T,S)=\Gamma(H,\Gamma(T,S)H_\alpha)$$

This result together with non-degeneracy of $\Gamma|_{\mathcal{H}\times\mathcal{H}}$ implies that $[T,S]=\Gamma(T,S)H_\alpha$. As in part (i) it can be shown that Γ is non-degenerate when restricted to $\mathcal{L}_\alpha\oplus\mathcal{L}_{-\alpha}$, so there must be a $T\in\mathcal{L}_\alpha$ and an $S\in\mathcal{L}_{-\alpha}$ with $\Gamma(T,S)\neq 0$. Hence, by suitably normalising T and S one can achieve $\Gamma(T,S)=1$.

(iv), (v) In line with (iii) choose $T\in\mathcal{L}_\alpha$, $S\in\mathcal{L}_{-\alpha}$ and $H_\alpha\in\mathcal{H}$ such that $[T,S]=H_\alpha$. Define the space

$$V=\mathbb{C}S+\mathbb{C}H_\alpha+\sum_{k\geq 1}\mathcal{L}_{k\alpha}$$

which is invariant under $\mathrm{ad}(H_\alpha)$. We prove the statement by computing the trace of $\mathrm{ad}(H_\alpha)|_V$ in two different ways. First, since the trace of a commutator always vanishes it follows that

$$\mathrm{tr}(\mathrm{ad}(H_\alpha)|_V)=\mathrm{tr}(\mathrm{ad}([T,S])|_V)=\mathrm{tr}([\mathrm{ad}(T)|_V,\mathrm{ad}(S)|_V])=0\ .$$

On the other hand, we can work out the trace directly by evaluating $\mathrm{ad}(H_\alpha)$ on a basis of V. This leads to

$$\begin{aligned}
\mathrm{ad}(H_\alpha)(S)&=[H_\alpha,S]=-\alpha(H_\alpha)S=-(\alpha,\alpha)S\\
\mathrm{ad}(H_\alpha)(H_\alpha)&=0\\
\mathrm{ad}(H_\alpha)(U)&=[H_\alpha,U]=k\alpha(H_\alpha)=k(\alpha,\alpha)U\quad\text{for}\quad U\in\mathcal{L}_{k\alpha}
\end{aligned}$$

which implies for the trace that

$$\mathrm{tr}(\mathrm{ad}(H_\alpha)|_V)=(\alpha,\alpha)\left(-1+\sum_{k\geq 1}k\ \dim(\mathcal{L}_{k\alpha})\right)\overset{!}{=}0\ .$$

The expression in the bracket can only be zero if $\dim(\mathcal{L}_\alpha)=1$ and $\dim(\mathcal{L}_{k\alpha})=0$ for $k>1$.

(vi) We need to work out the Killing form on two elements $H,\tilde{H}\in\mathcal{H}$, so we start with

$$\mathrm{ad}(H)\circ\mathrm{ad}(\tilde{H})(T)=[H,[\tilde{H},T]]=\alpha(H)\alpha(\tilde{H})T\ .$$

For the trace this implies

$$\Gamma(H, \tilde{H}) = \mathrm{tr}(\mathrm{ad}(H) \circ \mathrm{ad}(\tilde{H})) = \sum_{\alpha \in \Delta} \alpha(H)\alpha(\tilde{H})$$

which is the desired formula.

(vii) Assume that $\mathrm{Span}(\Delta) \neq \mathcal{H}'$, choose a basis $(\alpha_1, \ldots, \alpha_n)$ of $\mathrm{Span}(\Delta)$, complete to a basis $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_N)$ of $\mathcal{H}'$ and introduce the dual basis $(H_1, \ldots, H_N)$ of $\mathcal{H}$ so that $\alpha_i(H_j) = \delta_{ij}$. It follows that $\alpha(H_N) = 0$ for all $\alpha \in \Delta$ and, with the formula from (vi), that $\Gamma(H_N, H) = 0$ for all $H \in \mathcal{H}$. However, this contradicts the non-degeneracy of $\Gamma|_{\mathcal{H} \times \mathcal{H}}$. $\square$

Cartan-Weyl basis

Based on this theorem, we can now construct the **Cartan-Weyl basis** of $\mathcal{L}$. We start by choosing a basis (H_i), where $i = 1, \ldots, r = \mathrm{rk}(\mathcal{L})$, of the Cartan (this is, of course, not unique). Also, we know that the eigenspaces $\mathcal{L}_\alpha$ are one-dimensional for $\alpha \neq 0$ so we can choose $E_\alpha \in \mathcal{L}$ such that $\mathcal{L}_\alpha = \mathrm{Span}(E_\alpha)$. In addition, these can be normalised so that $\Gamma(E_\alpha, E_{-\alpha}) = 1$. In summary, we have the basis

$$(H_i, E_\alpha) \quad \text{where} \quad i = 1, \ldots, r = \mathrm{rk}(\mathcal{L}),\ \alpha \in \Delta \quad \text{and} \quad \Gamma(E_\alpha, E_{-\alpha}) = 1\ . \tag{4.26}$$

Relative to the basis (H_i) on the Cartan [1], elements $\alpha \in \mathcal{H}'$ can be described by a vector $(\alpha_1, \ldots, \alpha_r)$, where $\alpha_i = \alpha(H_i)$.

The Killing form, relative to the Cartan Weyl basis, then has components

$$\begin{array}{lll lll} \gamma_{ij} & := & \Gamma(H_i, H_j) = \sum_{\alpha \in \Delta} \alpha_i \alpha_j & \gamma_{i\alpha} & := & \Gamma(H_i, E_\alpha) = 0 \\ \gamma_{\alpha,-\alpha} & := & \Gamma(E_\alpha, E_{-\alpha}) = 1 & \gamma_{\alpha\beta} & := & \Gamma(E_\alpha, E_\beta) = 0\ , \quad \alpha + \beta \neq 0 \end{array} \tag{4.27}$$

Combining these results into a matrix leads to the block structure

$$[\Gamma] = \begin{pmatrix} & & & & & \cdots \\ (\gamma_{ij}) & & & & & \cdots \\ & & & & & \cdots \\ & 0 & 1 & & & \cdots \\ & 1 & 0 & & & \cdots \\ & & & 0 & 1 & \cdots \\ & & & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{array}{c} \\ \mathcal{H} \\ \\ \mathcal{L}_\alpha \\ \mathcal{L}_{-\alpha} \\ \mathcal{L}_\beta \\ \mathcal{L}_{-\beta} \\ \vdots \end{array},$$

with all entries outside the indicated blocks zero. We need to collect a few more pieces of information. First the Cartan element H_α, related to $\alpha \in \mathcal{H}'$ via Theorem 4.6 (i) must be a linear combination of the form $H_\alpha = \lambda^i H_i$. The short calculation

$$\alpha_j = \alpha(H_j) = \Gamma(H_\alpha, H_j) = \lambda^i \Gamma(H_i, H_j) = \lambda^i \gamma_{ij}$$

shows that $\lambda^i = \gamma^{ij}\alpha_j$ (where γ^{ij} is the inverse of γ_{ij}). We can use the Killing metric γ_{ij} on the Cartan and its inverse to lower and raise indices. In this spirit we write

$$H_\alpha = \alpha^i H_i \quad \text{where} \quad \alpha^i = \gamma^{ij}\alpha_j\ . \tag{4.28}$$

[1] Henceforth, we use indices $i, j, \ldots = 1, \ldots, r = \mathrm{rk}(\mathcal{H})$ to label the basis of the Cartan.

Using the same notation, the Killing form for $\alpha, \beta \in \mathcal{H}'$ can be written as

$$(\alpha, \beta) = \beta(H_\alpha) = \alpha^j \beta(H_j) = \alpha^j \beta_j = \gamma^{ij} \alpha_i \beta_j \; . \tag{4.29}$$

From Theorem 4.6 and Eqs. (4.22), (4.24), the commutation relations for the Cartan-Weyl basis take the form

$$\begin{array}{lclclcl} [H_i, H_j] & = & 0 & \qquad & [H_i, E_\alpha] & = & \alpha_i E_\alpha \\ [E_\alpha, E_{-\alpha}] & = & H_\alpha = \alpha^i H_i & \qquad & [E_\alpha, E_\beta] & = & \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{for} \quad 0 \neq \alpha + \beta \in \Delta \\ 0 & \text{for} \quad 0 \neq \alpha + \beta \notin \Delta \end{cases} \end{array} \; , \tag{4.30}$$

where $N_{\alpha\beta}$ are constants. An interesting and very helpful observation about these commutation relations is that for any root $\alpha \in \Delta$ the three generators $(H_\alpha, E_\alpha, E_{-\alpha})$ form a subalgebra with commutation relations

$$[H_\alpha, E_{\pm\alpha}] = \pm\alpha(H_\alpha) E_{\pm\alpha} = \pm(\alpha, \alpha) E_{\pm\alpha} \; , \qquad [E_\alpha, E_{-\alpha}] = H_\alpha \tag{4.31}$$

Comparison with Eq. (4.18) shows that this is, in fact, an $\mathrm{su}(2)_{\mathbb{C}}$ algebra (the only difference being the factor (α, α) which can be removed by a re-scaling of the three generators).

4.6 Weights

Basics of weights

Representations $r : \mathcal{L} \to \mathrm{End}(V)$ of a complex vector space V can be described following what we have done for the adjoint representation.

Definition 4.7. *For a representation $r : \mathcal{L} \to \mathrm{End}(V)$ we call $\omega \in \mathcal{H}'$ a* ***weight*** *of r if there is a non-zero vector $v \in V$ such that*

$$r(H)(v) = w(H)v \quad \textit{for all} \quad H \in \mathcal{H} \tag{4.32}$$

The ***eigenspace*** *of a weight w, denoted by V_ω, consists of all $v \in V$ which satisfy Eq. (4.32).*

In other words, we are looking for common eigenvectors of the Cartan, just as we have done for the adjoint representation. This means that the weights of the adjoint representation are the roots. The representation vector space V can be written as

$$V = \bigoplus_\omega V_\omega \; , \tag{4.33}$$

where the sum runs over all weights of the representation r. In practice, weights can be described by their values on the basis (H_i) of $\mathcal{H}$, so that a weight $w \in \mathcal{H}'$ is represented by the numerical vector $(w(H_1), \ldots, w(H_r))$, where $r = \mathrm{rk}(\mathcal{L})$.

Raising and lowering operators

How do the representation maps $r(E_\alpha)$ act on eigenvectors $v \in V_\omega$? A short calculation

$$\begin{aligned} r(H)\,(r(E_\alpha)v) &= [r(E_\alpha)r(H) + [r(H), r(E_\alpha)]]\,v = [\omega(H)r(E_\alpha) + r([H, E_\alpha])]\,v \\ &= [\omega(H)r(E_\alpha) + r(\alpha(H)E_\alpha)]\,v = [\omega(H)r(E_\alpha) + \alpha(H)r(E_\alpha)]\,v \\ &= [\omega(H) + \alpha(H)]\,r(E_\alpha)v \end{aligned} \tag{4.34}$$

shows that $r(E_\alpha)v \in V_{\omega+\alpha}$. So, applying E_α to a vector with weight ω leads to a vector with weight $\omega + \alpha$. This justifies thinking of the E_α as "raising" and "lowering" operators, in analogy with $E_\pm$ in the case of $\mathrm{su}(2)_\mathbb{C}$ (except we haven't quite decided yet which ones to call "raising" and which ones "lowering" - this is less straightforward in a multi-dimensional situation).

The above calculation suggests that weights in a representation differ by roots. This is made more precise in the following proposition.

Proposition 4.4. *If $r : \mathcal{L} \to \mathrm{End}(V)$ is irreducible then any two weights ω_1, ω_2 of r satisfy $\omega_1 - \omega_2 \in \Lambda_R$, that is, differences of weights are in the root lattice.*

Proof. If $\omega_1 - \omega_2 \notin \Lambda_R$ then $\bigoplus_{\alpha \in \Lambda_R} V_{\omega_1+\alpha} \subset V$ is invariant under r and genuinely smaller than V since it doesn't contain V_{ω_2}. Since r is irreducible this is excluded. $\square$

Weights of tensor representations

We should try to understand what happens to weights if we form the tensor product of two representations. To do this we should first work out what the tensor product means at the level of the Lie algebra. Consider two group representations $R_V : G \to \mathrm{GL}(V)$ and $R_{\tilde{V}} : G \to \mathrm{GL}(\tilde{V})$ of a Lie group G. These representations are related to their Lie algebra counterparts $r_V : \mathcal{L}(G) \to \mathrm{End}(V)$ and $r_{\tilde{V}} : \mathcal{L}(G) \to \mathrm{End}(\tilde{V})$ by

$$\left.\begin{array}{rcl} R_V(g) & = & \mathbb{1} + r_V(T) + \cdots \\ R_{\tilde{V}}(g) & = & \mathbb{1} + r_{\tilde{V}}(T) + \cdots \end{array}\right\} \quad \Rightarrow \quad (R_V \otimes R_{\tilde{V}})(g) = \mathbb{1} + r_V(T) \otimes \mathbb{1} + \mathbb{1} \otimes r_{\tilde{V}}(T) + \cdots \ ,$$

where $T \in \mathcal{L}(G)$ exponentiates to $g \in G$. We conclude that that the tensor representation at the algebra level is given by

$$r_{V\otimes\tilde{V}}(T) = r_V(T) \otimes \mathbb{1} + \mathbb{1} \otimes r_{\tilde{V}}(T) \ . \tag{4.35}$$

Now consider weights w and $\tilde{w}$ of r and $\tilde{r}$ with associated eigenvectors $v \in V$ and $\tilde{v} \in \tilde{V}$, so that $r_V(H)v = w(H)v$ and $r_{\tilde{V}}(H)\tilde{v} = \tilde{w}(H)\tilde{v}$. It follows that

$$r_{V\otimes\tilde{V}}(H)(v \otimes \tilde{v}) = (r_V(H)(v)) \otimes \tilde{v} + v \otimes (r_{\tilde{V}}(H)(\tilde{v})) = (w(H) + \tilde{w}(H))(v \otimes \tilde{v})$$

so the tensor state $v \otimes \tilde{v}$ has weight $w + \tilde{w}$. In short, weights add up under tensoring.

Weights of dual representations

What happens to weights if we pass from a representation $R : G \to \mathrm{GL}(V)$ to its dual representation $R' : G \to \mathrm{GL}(V')$? Writing $R(g) = \mathbb{1} + r(T)$ and $R'(g) = \mathbb{1} + r'(T)$, as before, the condition $R'(g)(\lambda))(R(g)(v)) = \lambda(v)$ which defines the dual representation translates into

$$(r'(T)(\lambda))(v) + \lambda(r(T)(v)) = 0 \ , \tag{4.36}$$

and this relation defines the dual representation r' of r at the Lie algebra level. This means that $r'(T) = -r(T)^T$, so if we set $T = H$ we learn that the weights are related by $\omega'(H) = -\omega(H)$. The weights of the dual representation are the negatives of the original weights.

Values of weights

Recall from Eq. (4.31) that $(H_\alpha, E_\alpha, E_{-\alpha})$ span an $su(2)_\mathbb{C}$ sub-algebra of $\mathcal{L}$ for all roots $\alpha \in \Delta$. Representations of this subalgebra have weights ω which evaluate to $\omega(H_\alpha)$ and, from the results on su$(2)_\mathbb{C}$ representations, these values must be in $\frac{(\alpha,\alpha)}{2}\mathbb{Z}$, where the additional factor of (α, α) accounts for the difference in normalisation between the algebras (4.31) and (4.18). In conclusion,

$$(\omega, \alpha) = \omega(H_\alpha) \in \frac{(\alpha, \alpha)}{2}\mathbb{Z} \quad \text{for all weights } \omega \text{ and all roots } \alpha \,. \tag{4.37}$$

We can formalise this statement by introducing the **weight lattice**

$$\Lambda_W = \left\{ \omega \in \mathcal{H}' \;\middle|\; \frac{2(\omega, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \;\forall\, \alpha \in \Delta \right\} \,. \tag{4.38}$$

It follows that all weights of a representation of $\mathcal{L}$ lie in Λ_W.

In the following, the idea is to construct irreps r of $\mathcal{L}$ by finding a procedure to determine the weight system of r. This idea is modelled on what we did for su$(2)_\mathbb{C}$: first find the highest weight $m = j$ and then apply the lowering operator, which changes the weights by the root -1, to find the entire weight system $j, j-1, \ldots, -j$. In the present case, the weights and roots are generally multi-dimensional so some additional twists and generalisations are in order. In one dimensions it is clear what is meant by lowering the weights. In higher dimensions, we have to define the difference between positive and negative roots.

Positive and negative roots

Choose a direction $\ell \in \mathcal{H}'$ in root space and define the two subsets of roots

$$\begin{array}{lcll} \Delta_+ & = & \{\alpha \in \Delta \,|\, \ell(\alpha) > 0\} & \text{(positive roots)} \\ \Delta_- & = & \{\alpha \in \Delta \,|\, \ell(\alpha) < 0\} & \text{(negative roots)}\,, \end{array} \tag{4.39}$$

where $\ell(\alpha) = \alpha(H_\ell)$. Of course we need to be slightly careful about the choice of ℓ so that $\ell(\alpha) \neq 0$ for all $\alpha \in \Delta$ but since Δ is a finite set of roots this is always possible. This granted we have $\Delta = \Delta_+ \cup \Delta_-$ and $\Delta_- = -\Delta_+$. The E_α with $\alpha \in \Delta_+$ are considered the raising operators and the E_α with $\alpha \in \Delta_-$ the lowering operators. Given this distinction we can now define the highest weight and highest weight vector.

Definition 4.8. *Let $r : \mathcal{L} \to \mathrm{End}(V)$ be a representation. A non-zero vector $v \in V$ is called a **highest weight vector** of r if $E_\alpha(v) = 0$ for all $\alpha \in \Delta_+$. The weight λ of a highest weight vector is called **highest weight**.*

Of course we should worry about existence and uniqueness of the highest weight vector.

Lemma 4.3. *For a semi-simple complex Lie algebra $\mathcal{L}$ with representation $r : \mathcal{L} \to \mathrm{End}(V)$*

(i) *r has a highest weight vector $v \in V$.*

(ii) *Successive application of E_α, where $\alpha \in \Delta_-$ on v gives a sub-representation of r. If r is an irrep it is obtained in this way.*

(iii) *If r is an irrep the highest weight vector is unique up to re-scaling.*

Proof. (i) Define the highest weight λ as the weight for which $\ell(\lambda) = \lambda(H_\ell)$ is maximal and choose a vector $v \in V_\lambda$. For all $\alpha \in \Delta_+$ we have $\ell(\lambda+\alpha) = \ell(\lambda) + \ell(\alpha) > \ell(\lambda)$ but since $\ell(\lambda)$ was the maximum over all weights this means that $\lambda + \alpha$ cannot be a weight, so $V_{\lambda+\alpha} = \{0\}$. Hence $E_\alpha v = 0$ which shows that v is a highest weight vector.

(ii) We begin by defining $W_k = \mathrm{Span}\{E_{\alpha_1} \cdots E_{\alpha_k} v \,|\, \alpha_i \in \Delta_-\}$ and $W = \bigoplus_k W_k \subset V$. This means that W is the subspace of V which is spanned by all the vectors obtained by acting with lowering operators on the highest weight vector - exactly the type of construction we have envisaged earlier. We need to show that W is invariant under all generators. This is obvious for lowering operators E_α, where $\alpha \in \Delta_-$ since $E_\alpha W_k \subset W_{k+1}$.

For an element $H \in \mathcal{H}$ in the Cartan we have

$$\begin{aligned} HE_{\alpha_1}E_{\alpha_2}\cdots E_{\alpha_k}v &= (E_{\alpha_1}H + [E_{\alpha_1}, H])E_{\alpha_2}\cdots E_{\alpha_k}v \\ &= E_{\alpha_1}HE_{\alpha_2}\cdots E_{\alpha_k}v - \alpha_1(H)E_{\alpha_1}E_{\alpha_2}\cdots E_{\alpha_k}v \end{aligned}$$

By repeating the above commutation step H can be moved further and further to the right, picking up a term proportional to $E_{\alpha_1}E_{\alpha_2}\cdots E_{\alpha_k}v$ each time until H is next to v in which case we use $Hv = \lambda(H)v$. Altogether, this shows that $H(W_k) \subset W_k$.

Finally, we need to worry about raising operators E_α, where $\alpha \in \Delta_+$. We claim that $E_\alpha W_k \subset \bigoplus_{i\le k} W_i$ and show this by induction in k. For $k = 0$ we have $E_\alpha W_0 = E_\alpha \mathrm{Span}(v) = \{0\} \subset W_0$. Now assume the statement is true for $k-1$ and work out

$$E_\alpha E_{\alpha_1}E_{\alpha_2}\cdots E_{\alpha_k}v = E_{\alpha_1}\overbrace{E_\alpha \underbrace{E_{\alpha_2}\cdots E_{\alpha_k}v}_{\in W_{k-1}}}^{\in\bigoplus_{i\le k-1}W_i \text{ by ind.}} + \overbrace{\underbrace{[E_\alpha, E_{\alpha_1}]}_{\in\mathcal{H}\text{ or }\sim E_{\alpha+\alpha_1}}\ E_{\alpha_2}\cdots E_{\alpha_k}v}^{\in W_k\cup W_{k-1}} \in \bigoplus_{i\le k} W_i$$

This completes the induction step and we have shown that $W \subset V$ carries a representation. If W was a direct sum, so $W = W_1 \oplus W_2$, then $v \in W_1$ or $v \in W_2$ but then repeating the above construction would lead to the same set of vectors so either $W_1 = W$ or $W_2 = W$.

(iii) Suppose there are two linearly independent highest weight vectors $v_1, v_2 \in V_\lambda$. Then applying the procedure in (ii) to v_1 generates an irrep which does not contain v_2, so that V is not irreducible. Hence, $\dim(V_\lambda) = 1$. □

Simple roots

The previous Lemma describes a practical, algorithmic way of constructing a representation starting with a highest weight and a highest weight vector, by successively applying lowering operators to the highest weight. If α, β and $\alpha+\beta$ are weights we know from the commutation relations that $E_{\alpha+\beta}$ is proportional to $[E_\alpha, E_\beta]$. This means, in the above algorithm to sweep out an irrep we only need to consider roots which are not sums of two other roots.

Definition 4.9. *A positive (negative) root is called **simple** if it cannot be written as a sum of two positive (negative) roots.*

Corollary 4.2. *For simple roots we have the following statements.*

(i) *An irrep r can be generated by successively applying lowering operators E_α to the highest weight vector v, where α is a simple negative root.*

(ii) *The simple positive (or negative) roots form a basis of $\mathcal{H}'$.*

Proof. (i) This follows from part (ii) of the previous Lemma and the fact that $E_{\alpha+\beta}$ is proportional to $[E_\alpha, E_\beta]$.

(ii) We know from Theorem 4.6 (vii) that Δ contains a basis of $\mathcal{H}'$. Since $\Delta_- = -\Delta_+$ either of $\Delta_\pm$ must already contain a basis of $\mathcal{H}'$ and since the simplicity requirement eliminates linearly dependent vectors this is still true for the positive (or negative) simple roots. □

Dynkin labels and Cartan matrix

Courtesy of part (ii) of the above lemma we can introduce a basis $(\alpha_1, \dots, \alpha_r)$, where $r = \mathrm{rk}(\mathcal{L})$, of $\mathcal{H}'$ which consists of positive simple roots. For any weight $w \in \mathcal{H}'$ we can then define the quantities

$$a_i = \frac{2(w, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z} \quad \text{where} \quad i = 1, \dots, r\ . \tag{4.40}$$

Since all weights are in the weight lattice (4.38) these quantities are indeed integers. The vector $(a_1, \dots, a_r)$ of these integers characterises the weight w and it is called the **Dynkin label** of the weight w.

The $r \times r$ matrix whose rows are the Dynkin labels of the positive simple roots is called the **Cartan matrix** of $\mathcal{L}$ and it is explicitly given by

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}\ . \tag{4.41}$$

As we will see, the Cartan matrix encodes all the relevant information about the Lie algebra and it can be used to reconstruct the entire root system and the weight system of all irreps. To accomplish the latter we require a statement about the highest weights of irreps.

Theorem 4.7. *The highest weights of all irreps of $\mathcal{L}$ are given by Dynkin labels $(a_1, \dots, a_r)$ with all $a_i \geq 0$.*

Proof. See Ref. [1], App. D. □

With these statements we now have an outline of how to proceed systematically in the quest to understand semi-simple complex Lie algebras $\mathcal{L}$ and their representations. The classification of these algebras can, in fact, be accomplished by thinking about the possible Cartan matrices and we will work this out later. With such a classification in hand, we choose the algebra, that is, a Cartan matrix. Then we select a highest weight Dynkin label $(a_1, \dots, a_r)$ for the irrep we would like to construct and apply the procedure outlined in Lemma 4.3 to obtain the full system of weights. We will develop the details of this process later.

Chapter 5

Examples of Lie groups

It is now time to put the results so far to some use and discuss examples beyond the cases of SU(2) and SO(3). These include the Lorentz and Poincaré groups in four dimensions, the unitary groups and the orthogonal groups.

5.1 The Lorentz group

The Lorentz group in four dimensions is of pre-eminent importance in physics, since it is an underlying symmetry for all relativistic theories. Its group structure and its representation theory determines the basic rules for building relativistic (quantum) field theories.

Definition of Lorentz group

The **Lorentz group** $L = O(3,1)$ is the group of real 4×4 matrices which leaves the **Lorentz metric** $\eta = \mathrm{diag}(-1,1,1,1)$ invariant, so

$$L = O(3,1) = \left\{ \Lambda \in \mathrm{GL}(\mathbb{R}^4) \mid \Lambda^T \eta \Lambda = \eta \right\} \tag{5.1}$$

For index notation, we will usually use indices $\mu, \nu = 0,1,2,3$ for space-time, and we write $\Lambda = (\Lambda^\mu{}_\nu)$ and $\eta = (\eta_{\mu\nu})$, while indices $i, j, \ldots = 1,2,3$ are used for the three spatial directions. The defining relation of the Lorentz group in index notation is

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad \Leftrightarrow \quad \Lambda_\mu{}^\rho \Lambda^\nu{}_\rho = \delta^\nu_\mu \tag{5.2}$$

Note that the three-dimensional orthogonal group O(3) can be thought of as a subgroup of the Lorentz group via the embedding

$$\mathrm{O}(3) \ni R \mapsto \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \in L \; . \tag{5.3}$$

Global structure

Taking the determinant of the defining relation $\Lambda^T \eta \Lambda = \eta$ it follows that $\det(\Lambda) \in \{\pm 1\}$ for any $\Lambda \in L$. There is another sign choice, as can be seen from the following calculation, focusing on the (00) component of Eq. (5.2).

$$\eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = -1 \quad \Rightarrow \quad (\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1 \quad \Rightarrow \quad \Lambda^0{}_0 \geq 1 \text{ or } \Lambda^0{}_0 \leq -1 \; .$$

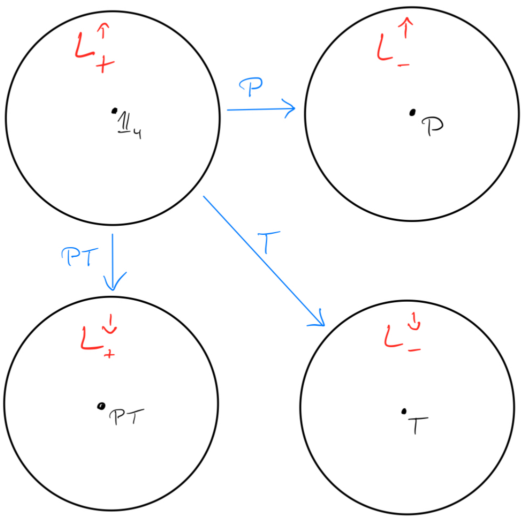
Altogether, we have four combinations of sign choices for $\det(\Lambda)$ and $\Lambda^0{}_0$ and with the specific Lorentz group elements

$$P = \mathrm{diag}(1,-1,-1,-1) \quad \text{(parity)}\ , \qquad T = \mathrm{diag}(-1,1,1,1) \quad \text{(time inversion)}\ , \tag{5.4}$$

we see that all four possible sign combinations are in fact realised since $\{\mathbb{1}_4, P, T, PT = -\mathbb{1}_4\} \subset L$. More generally, the Lorentz group consists of four path-disconnected pieces, corresponding to the four sign combinations, which are defined as

$$\begin{array}{rcll} L_+^\uparrow &=& \{\Lambda \in L \mid \det\Lambda = 1, \Lambda_0^0 \geq 1\} \ni \mathbb{1} & \text{(proper orthochronous)} \\ L_+^\downarrow &=& \{\Lambda \in L \mid \det\Lambda = 1, \Lambda_0^0 \leq -1\} = PTL_+^\uparrow & \\ L_-^\uparrow &=& \{\Lambda \in L \mid \det\Lambda = -1, \Lambda_0^0 \geq 1\} = PL_+^\uparrow & \\ L_-^\downarrow &=& \{\Lambda \in L \mid \det\Lambda = -1, \Lambda_0^0 \leq -1\} = TL_+^\uparrow & \end{array} \tag{5.5}$$

and are each generated from the **proper orthochronous Lorentz group** $L_+^\uparrow$ by multiplication with one of the four group elements in $\{\mathbb{1}_4, P, T, PT = -\mathbb{1}_4\} \subset L$. The **proper Lorentz group** $\mathrm{SO}(3,1)$ consists of the Lorentz transformations with determinant one, so $\mathrm{SO}(3,1) = L_+^\uparrow \cup L_+^\downarrow$.



The Lorentz transformations routinely used in Special Relativity are contained in the proper orthochronous Lorentz group $L_+^\uparrow$ and they include the standard two-dimensional transformations

$$\Lambda = \mathrm{diag}(\Lambda_2, \mathbb{1}_2) \in L_+^\uparrow \quad \text{where} \quad \Lambda_2 = \begin{pmatrix} \cosh(\xi) & \sinh(\xi) \\ \sinh(\xi) & \cosh(\xi) \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\ , \tag{5.6}$$

where $\xi \in \mathbb{R}$ is called **rapidity**, $\beta = \tanh(\xi)$ is interpreted as the relative velocity of the two inertial systems in the x-direction and $\gamma = 1/\sqrt{1-\beta^2}$. The proper orthochronous Lorentz group $L_+^\uparrow$ also contains the three-dimensional rotations $\mathrm{SO}(3)$, via the embedding (5.3).

Application 5.1: *(Lorentz invariance in physics)*

When we talk about Lorentz invariance it is important to be clear about which of the four parts of the Lorentz group we have in mind. At the minimal level Lorentz invariance refers to invariance under the proper orthochronous Lorentz group $L_+^\uparrow$ and this is the sense in which the term is most commonly used in physics. If a theory is Lorentz invariant in this sense whether it is also invariant under the other parts of the Lorentz group comes down to the question of whether it is invariant under parity P and time inversion T (as is clear from Eqs. (5.5)). While the known relativistic theories of physics are invariant under $L_+^\uparrow$, invariance under P and T depends on the theory in question. While the theory of strong interactions (QCD) is P-invariant (and T-invariant at least to very high accuracy) the theory of weak interactions respects neither P nor T.

Lie algebra of the Lorentz group

The Lie algebra of the Lorentz group is the tangent space at the identity $\mathbb{1}_4$ which is contained in the proper orthochronous component $L_+^\uparrow$. For this reason, the Lie algebra carries no information about the other three components. We can determine the Lie algebra as usual, by writing down $\Lambda = 1 + T + \cdots$ and evaluating this to linear order on the defining relation

$$\Lambda^T \eta \Lambda = (\mathbb{1}_4 + T^T + \cdots)\eta(\mathbb{1}_4 + T + \cdots) = \eta + T^T\eta + \eta T + \cdots \stackrel{!}{=} \eta \quad \Rightarrow \quad T = -\eta T^T \eta$$

so that the Lie algebra is

$$\mathcal{L}(L) = \left\{T \in \mathrm{End}(\mathbb{R}^4) \mid T = -\eta T^T \eta\right\} \qquad \Rightarrow \qquad \dim(\mathcal{L}(L)) = 6\;. \tag{5.7}$$

The condition $T = -\eta T^T \eta$ is a mixed symmetry/anti-symmetry condition, depending on the signs in η, and it leads to anti-symmetry in the space-space entries of T and symmetry in the space-time entries (as well as to zero along the diagonal). There are at least two standard ways to write down a basis for $\mathcal{L}(L)$, namely

$$\mathcal{L}(L) = \mathrm{Span}(\sigma_{\mu\nu}) = \mathrm{Span}(\tilde{J}_i, \tilde{K}_i) \tag{5.8}$$

where $\sigma_{\mu\nu}$ are six matrices, labelled by a pair $(\mu\nu)$ of anti-symmetric indices and defined by

$$(\sigma_{\mu\nu})^\rho{}_\sigma = \eta^\rho_\mu \eta_{\nu\sigma} - \eta_{\mu\sigma}\eta^\rho_\nu\;. \tag{5.9}$$

Their commutation relations can be worked out by straightforward calculation from the above expression and are given by

$$[\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = \eta_{\alpha\delta}\sigma_{\beta\gamma} + \eta_{\alpha\gamma}\sigma_{\delta\beta} + \eta_{\beta\delta}\sigma_{\gamma\alpha} + \eta_{\beta\gamma}\sigma_{\alpha\delta}\;. \tag{5.10}$$

The advantage of this basis is that it is labelled by a pair of "covariant" four-dimensional indices. The other basis consists of essentially the same matrices but with the labelling broken up into the space-space and space-time components, according to

$$\tilde{J}_i = \frac{1}{2}\epsilon_{ijk}\sigma_{jk}\;, \qquad \tilde{K}_i = \sigma_{0i}\;. \tag{5.11}$$

More explicitly, these matrices are given by

$$\tilde{J}_i = \begin{pmatrix} 0 & 0 \\ 0 & T_i \end{pmatrix}\;, \qquad \tilde{K}_i \stackrel{i=1}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.12}$$

where T_i are the SO(3) generators from Eq. (3.39). The appearance of the SO(3) generators is of course not at all surprising, given the embedding of SO(3) into the Lorentz group, and they exponentiate to rotations of the form (5.3).The other three generators $\tilde{K}_i$ are called the **boost generators** and they exponentiate to genuine Lorentz transformations, such as those in Eq. (5.6). The commutators (5.10) can be translated to the $(\tilde{J}_i, \tilde{K}_i)$ basis and this leads to

$$[\tilde{J}_i, \tilde{J}_j] = \epsilon_{ij}{}^k \tilde{J}_k \ , \qquad [\tilde{K}_i, \tilde{K}_j] = -\epsilon_{ij}{}^k \tilde{J}_k \ , \qquad [\tilde{J}_i, \tilde{K}_j] = \epsilon_{ij}{}^k \tilde{K}_k \tag{5.13}$$

Evidently, the $\tilde{J}_i$ span an so(3) subalgebra of $\mathcal{L}(L)$.

The group $SL(2,\mathbb{C})$

We have seen earlier that the groups SO(3) and SU(2) are closely related. With SO(3) a subgroup of the Lorentz group it seems reasonable to ask if this relationship can be extended to L and, if so, what the required generalisation of $SU(2)$ would be. This group needs to contain SU(2) as a subgroup and its dimension needs to be six (the same as the Lorentz group's). It turns out that the correct group is the **special linear group** in two dimensions. The special linear group in n dimensions consists of complex matrices with determinant one, so

$$\mathrm{SL}(n,\mathbb{C}) = \{M \in \mathrm{GL}(\mathbb{C}^n) \,|\, \mathrm{det}M = 1\} \ . \tag{5.14}$$

The idea is now that the Lorentz group is a representation of $\mathrm{SL}(2,\mathbb{C})$, in the same way SO(3) is the adjoint of SU(2). To check this, we introduce the space $\mathcal{S} = \{S \in \mathrm{End}(\mathbb{C}^2) \,|\, S = S^\dagger\}$ of hermitian 2×2 matrices and the map

$$\mathcal{T} : \mathbb{R}^4 \xrightarrow{\cong} \mathcal{S} \ , \qquad \mathcal{T}(X) = X^\mu \sigma_\mu \ , \tag{5.15}$$

where $\sigma_\mu = (\mathbb{1}_2, \sigma_i)$ forms a basis of $\mathcal{S}$. The map $\mathcal{T}$ identifies a four-vector X^μ with the hermitian matrix

$$\mathcal{T}(X) = \begin{pmatrix} X^0 + X^3 & X^1 - iX^2 \\ X^1 + iX^2 & X^0 - X^3 \end{pmatrix} \ . \tag{5.16}$$

An interesting feature of this map, which is the analogue of Eq. (3.44), is that the Minkowski product of a vector can be expressed as

$$X^T \eta X = -\mathrm{det}(\mathcal{T}(X)) \ , \tag{5.17}$$

as can be quickly checked by working out the determinant of Eq. (5.16).

The crucial step is to introduce the $\mathrm{SL}(2,\mathbb{C})$ representation $R_V : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathbb{R}^4)$ by

$$R_V(M)(X) = \mathcal{T}^{-1}(M\mathcal{T}(X)M^\dagger) \quad \Leftrightarrow \quad \sigma_\mu R_V(M)^\mu{}_\nu X^\nu = M(X^\mu \sigma_\mu)M^\dagger \tag{5.18}$$

Note this makes sense: $\mathcal{T}(X)$ is a hermitian matrix, as is $M\mathcal{T}(X)M^\dagger$ which is converted back to a four-vector by $\mathcal{T}^{-1}$ - hence $R_V(M)$ does indeed map four-vectors to four-vectors. How does $R_V(M)$ relate to the Minkowski product?

$$\begin{aligned}(R_V(M)(X))^T\eta(R_V(M)(X)) &\overset{(5.17)}{=} -\mathrm{det}(\mathcal{T}(R_V(M)(X))) \overset{(5.18)}{=} -\mathrm{det}(M\mathcal{T}(X)M^\dagger) \\ &\overset{\mathrm{det}(M)=1}{=} -\mathrm{det}(\mathcal{T}(X)) = X^T\eta X \ .\end{aligned}$$

This means $R_V(M)$ leaves the Minkowski product invariant and must, hence, be a Lorentz transformation. Since $\mathrm{SL}(2,\mathbb{C})$ is, in fact, path-connected, $R_V(M) \in L_+^\uparrow$. Much as for the analogous $SU(2)$ case, we can show that $\mathrm{Ker}(R_V) = \{\pm \mathbb{1}_2\}$ and that the image of R_V is all of $L_+^\uparrow$. Altogether, this means that

$$L_+^\uparrow \cong \frac{\mathrm{SL}(2,\mathbb{C})}{\mathbb{Z}_2} \; , \tag{5.19}$$

which is of course the analogue of Eq. (3.46).

Lie algebra of $\mathrm{SL}(2,\mathbb{C})$

The isomorphism (5.19) implies an isomorphism between the corresponding Lie algebras and to see this explicitly we should first work out the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$. With $M = \mathbb{1}_2 + T + \cdots$, the condition $\det(M) = 1$ translates into $\mathrm{tr}(T) = 0$, so

$$\mathrm{sl}(2,\mathbb{C}) = \mathcal{L}(\mathrm{SL}(2,\mathbb{C})) = \{T \in \mathrm{End}(\mathbb{C}^2) \,|\, \mathrm{tr}(T) = 0\} \quad \Rightarrow \quad \dim(\mathrm{sl}(2,\mathbb{C})) = 6 \; . \tag{5.20}$$

This Lie algebra consists of all traceless, complex 2×2 matrices and, since each such matrix can be written as a sum of a hermitian and an anti-hermitian traceless matrix, a basis for $\mathrm{sl}(2,\mathbb{C})$ can be easily written down in terms of Pauli matrices:

$$J_i = -i\frac{\sigma_i}{2} \; , \qquad K_i = -\frac{\sigma_i}{2} \tag{5.21}$$

Their commutation relations follow easily from the commutators (3.34) of the Pauli matrices and are given by

$$[J_i, J_j] = \epsilon_{ij}{}^k J_k \; , \quad [K_i, K_j] = -\epsilon_{ij}{}^k J_k \; , \quad [J_i, K_j] = \epsilon_{ij}{}^k K_k \; . \tag{5.22}$$

The generators J_i span the $\mathrm{su}(2)$ subalgebra of $\mathrm{sl}(2,\mathbb{C})$. The linear map $\mathrm{sl}(2,\mathbb{C}) \to \mathcal{L}(L)$ defined by $J_i \mapsto \tilde{J}_i$ and $K_i \mapsto \tilde{K}_i$ is a Lie algebra isomorphism, as comparison of Eqs. (5.13) and (5.22) shows.

Representations

The commutation relations (5.22) not only contain the relations for $\mathrm{su}(2)$ but, as a whole, are quite reminiscent of the $\mathrm{su}(2)$ relations. This can be made more explicit by defining the new $\mathrm{sl}(2,\mathbb{C})$ basis

$$J_i^\pm = \frac{1}{2}(J_i \pm iK_i) \; , \tag{5.23}$$

with commutation relations (derived from Eq. (5.22))

$$[J_i^\pm, J_j^\pm] = \epsilon_{ij}{}^k J_k^\pm \; , \quad [J_i^+, J_j^-] = 0 \qquad \Rightarrow \qquad \mathrm{sl}(2,\mathbb{C}) \cong \mathrm{su}(2) \oplus \mathrm{su}(2) \; . \tag{5.24}$$

It is a lucky coincidence that the Lorentz group algebra is isomorphic to two copies of $\mathrm{su}(2)$. This means that studying its representation does not require the full machinery of Lie group representations (as more complicated groups will) but we can rely on our knowledge of $\mathrm{su}(2)_\mathbb{C}$ representations.

We know that $\mathrm{su}(2)_\mathbb{C}$ representations are classified by a "spin" $j \in \mathbb{Z}/2$ and have dimension $2j + 1$, so $\mathrm{sl}(2,\mathbb{C})$ representations are classified by two spins $(j_+, j_-) \in \mathbb{Z}/2 \times \mathbb{Z}/2$ and have dimension $(2j_+ + 1)(2j_- + 1)$. More explicitly, if $r_\pm$ are the $\mathrm{su}(2)_\mathbb{C}$ representations with spin

$j_\pm$ then the corresponding $\mathrm{sl}(2,\mathbb{C})$ representation $r_{(j_+,j_-)}:\mathrm{sl}(2,\mathbb{C})\to\mathrm{End}(V_{(j_+,j_-)})$ (where $V_{(j_+,j_-)}\cong\mathbb{C}^{(2j_++1)(2j_-+1)})$) has dimension

$$\dim(r_{(j_+,j_-)})=(2j_++1)(2j_-+1)\;, \tag{5.25}$$

and is explicitly specified by

$$r_{(j_+,j_-)}(J_i^+)=r_+(J_i^+)\times\mathbb{1}_{2j_-+1}\;,\qquad r_{(j_+,j_-)}(J_i^-)=\mathbb{1}_{2j_++1}\times r_-(J_i^-)\;. \tag{5.26}$$

By inverting Eq. (5.23) the representation matrices of the original generators are then

$$\begin{array}{rcl} r_{(j_+,j_-)}(J_i) & = & r_+(J_i^+)\times\mathbb{1}_{2j_-+1}+\mathbb{1}_{2j_++1}\times r_-(J_i^-) \\ r_{(j_+,j_-)}(K_i) & = & -i\left(r_+(J_i^+)\times\mathbb{1}_{2j_-+1}-\mathbb{1}_{2j_++1}\times r_-(J_i^-)\right) \end{array}\;. \tag{5.27}$$

We can lift this up to the group level using the exponential map (writing $r=r_{(j_+,j_-)}$ and R for its counterpart at the group level):

$$M=\exp(t^iJ_i+s^iK_i)\quad\mapsto\quad R(M)=\exp(t^ir(J_i)+s^ir(K_i))\;.$$

Examples of $\mathrm{sl}(2,\mathbb{C})$ representations

The $\mathrm{sl}(2,\mathbb{C})$ representations with low dimensions play an important role in relativistic physics. They classify the types of particles - scalars, fermions, vector fields - which arise and they determine the structure of relativistic theories. For this reason, we work out these simplest representations explicitly. (For simplicity of notation, we will drop the representation r in the following and, for example, write J_i^+ instead of $r(J_i^+)$ - which representation r is referred to will be clear from the context.)

$(j_+,j_-)=(0,0)$ *(Scalars)*

From Eq. (5.25), this is a one-dimensional representation on $V_{(0,0)}\cong\mathbb{C}$. Since it is built from two spin zero representations we have to choose $J_i^\pm=0$ which, from Eq. (5.27), implies $J_i=K_i=0$. From Eq. (5.1) this means $R(M)=1$ for all $M\in\mathrm{SL}(2,\mathbb{C})$, so this is the trivial representation. In physics this is also called the **scalar representation** of the Lorentz group and fields which take values in $V_{(0,0)}$ are referred to as **scalar fields**.

$(j_+,j_-)=(1/2,0)$ *(Left-handed Weyl spinors)*

From Eq. (5.25) this is a two-dimensional representation, $V_{(1/2,0)}\cong\mathbb{C}^2$, and we have to choose $J_i^+=-i\sigma_i/2$ (the standard generators in the fundamental of $\mathrm{su}(2)_\mathbb{C}$) and $J_i^-=0$ (the right choice for spin zero) which, via Eq. (5.27), translates to $J_i=-i\sigma_i/2$ and $K_i=-\sigma_i/2$. Exponentiating, using Eq. (5.1) leads to

$$R_L(M)=R_{(1/2,0)}(M)=\exp\left[\frac{1}{2}(-s^i-it^i)\sigma_i\right]\;. \tag{5.28}$$

This representation is called the **left-handed Weyl spinor representation** and fields taking values in $V_{(1/2,0)}$ are called **left-handed Weyl spinors**.

$(j_+,j_-)=(0,1/2)$ *(Right-handed Weyl spinors)*

From Eq. (5.25) this is a two-dimensional representation, $V_{(0,1/2)}\cong\mathbb{C}^2$, and we have to choose $J_i^+=0$ and $J_i^-=-i\sigma_i/2$ which, via Eq. (5.27), translates to $J_i=-i\sigma_i/2$ and $K_i=\sigma_i/2$. Exponentiating, using Eq. (5.1), leads to

$$R_R(M)=R_{(0,1/2)}(M)=\exp\left[\frac{1}{2}(s^i-it^i)\sigma_i\right]\;. \tag{5.29}$$

This representation is called the **right-handed Weyl spinor representation** and fields taking values in $V_{(0,1/2)}$ are called **right-handed Weyl spinors**. Note that this representation, as well as its left-handed counterpart, is not unitary (the argument of the exponent is in general not anti-hermitian).

$(1/2,0)\oplus(0,1/2)$ *(Dirac spinors)*

This is a four-dimensional (reducible under $\mathrm{sl}(2,\mathbb{C})$) representation, $R_D = R_L \oplus R_R$, built as a direct sum of the left- and right-handed Weyl spinors with representation matrices

$$R_D(M) = \begin{pmatrix} R_L(M) & 0 \\ 0 & R_R(M) \end{pmatrix} \tag{5.30}$$

This is called the **Dirac spinor representation** and fields taking values in $V_{(1/2,0)} \oplus V_{(0,1/2)} \cong \mathbb{C}^4$ are called **Dirac spinors**.

$(j_+, j_-) = (1/2, 1/2)$ *(Vector fields)*

From Eq. (5.25) this representation is four-dimensional and, on dimensional grounds (plus a parity argument, see below) it must be equal to the fundamental representation R_V of the Lorentz group L. Hence, the representation matrices can be written as

$$\Lambda(M) = R_V(M) = \exp(t^i \tilde{J}_i + s^i \tilde{K}_i) \tag{5.31}$$

with the Lorentz group generators $\tilde{J}_i, \tilde{K}_i$ from Eq. (5.11).

Parity

So far, we have studied representations of the Lie algebra which means representations of $L_+^\uparrow$, the component of the Lorentz group which contains the identity. For representations of the full Lorentz group L we have to consider the effect of parity $P = \mathrm{diag}(1,-1,-1,-1)$ on the representations. Since

$$P\tilde{J}_i P = \tilde{J}_i \ , \quad P\tilde{K}_i P = -\tilde{K}_i \qquad \Rightarrow \qquad PJ_i^\pm P = J_i^\mp$$

it follows that parity maps representations as $P : R_{(j_+,j_-)} \to R_{(j_-,j_+)}$, that is, it exchanges the two spins which characterise representations of $\mathrm{sl}(2,\mathbb{C})$. This means the only $\mathrm{sl}(2,\mathbb{C})$ irreps which are representations of the full Lorentz group are the representations $R_{(j_+,j_-)}$ with $j_+ = j_-$. From the above list, this includes only the vector representation $(j_+, j_-) = (1/2, 1/2)$. However, direct sums $R_{(j_+,j_-)} \oplus R_{(j_-,j_+)}$ also form representations under the full Lorentz group. They are reducible as $\mathrm{sl}(2,\mathbb{C})$ representations but irreducible as representations of the full Lorentz group. The Dirac spinor, $(1/2,0)\oplus(0,1/2)$, is an example.

Dual and complex conjugate of spinors

We can consider the dual and the complex conjugate representation and we would like to do this for the Weyl spinors constructed above. This will lead to useful identities which facilitate writing down Lorentz invariants. Since performing the dual and the complex conjugation lead to representations of the same dimensions and the left- and right-handed Weyl spinors are the only two-dimensional irreps we know that these two operations must relate the Weyl spinor the (complex conjugate) dual of the left-handed Weyl spinor we have

$$(R_L(M)^{-1})^\dagger \overset{(5.28)}{=} \exp\left[\frac{1}{2}(s^i - it^i)\sigma_i\right] \overset{(5.29)}{=} R_R(M)\ . \tag{5.32}$$

The dual of the left-handed Weyl spinor is the right-handed Weyl spinor (and vice versa, of course). For the complex conjugate we find

$$R_L(M)^* \stackrel{(5.28)}{=} \exp\left[\frac{1}{2}(-s^i+it^i)\sigma_i^*\right] \stackrel{(3.37)}{=} \sigma_2 \exp\left[\frac{1}{2}(s^i-it^i)\sigma_i\right]\sigma_2 \stackrel{(5.29)}{=} \sigma_2 R_R(M)\sigma_2^{-1} \quad (5.33)$$

The complex conjugate of the left-handed Weyl spinor is equivalent to the right-handed Weyl spinor via a basis transformation with σ_2. Taking the inverse complex conjugate of Eq. (5.32) and using Eq. (5.33) we also have

$$R_L(M)^T = (R_R(M)^{-1})^* = \sigma_2 R_L(M)^{-1}\sigma_2 \; . \quad (5.34)$$

Another, very useful relation follows from Eq. (5.18) (by dropping X, identifying M with $R_L(M)$ and writing $\Lambda = R_V(M)$):

$$R_R(M)^\dagger \sigma_\mu R_R(M) = \Lambda_\mu{}^\nu \sigma_\nu \quad \stackrel{\text{c.c.}}{\Longrightarrow} \quad R_L(M)^\dagger \bar{\sigma}_\mu R_L(M) = \Lambda_\mu{}^\nu \bar{\sigma}_\nu \; , \quad (5.35)$$

where $\bar{\sigma}_\mu = \sigma_2\sigma_\mu^*\sigma_2 = (\mathbb{1}_2, -\sigma_i)$.

Application 5.2: *(Lorentz invariant model building - a rough guide)*

The main players of Lorentz invariant field theories are fields which take values in the simple Lorentz group representations constructed above, so scalars, left- and right-handed Weyl spinors, Dirac spinors and vector fields. All fields are functions of space-time coordinates x^μ, where $\mu = 0, 1, 2, 3$, which transform as vectors, so that $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ and $\partial_\mu \mapsto \Lambda_\mu{}^\nu \partial_\nu$, where $\partial_\mu = \partial/\partial x^\mu$. Vectors carry a single μ index, tensors of the vector representation multiple indices $\mu, \nu, \ldots$ and, as a result of Eq. (5.2), we can construct invariants of these objects simply by contracting upper with lower indices.

Scalars $\phi = (\phi^a)$ (where a labels different scalars) with $\phi^a(x) \in V_{(0,0)}$ are Lorentz singlets and a general Lorentz group invariant Lagrange density is of the from

$$\mathcal{L}_{\text{scalar}} = -G_{ab}(\phi)\partial_\mu\phi_a\partial^\mu\phi^b - V(\phi) \; , \quad (5.36)$$

where G_{ab} and V are functions of ϕ, called the **field-space metric** and the **scalar potential**, respectively. The first term above is called the **kinetic term** - note that the contraction of the μ indices ensures it is Lorentz invariant.

We denote left- and right-handed Weyl spinors by $\chi_{L,R}$ or $\psi_{L,R}$, so that $\chi_L(x), \psi_L(x) \in V_{(1/2,0)}$ and $\chi_R(x), \psi_R(x) \in V_{(0,1/2)}$. This implies the Lorentz transformations

$$\chi_{L,R} \mapsto R_{L,R}(M)\chi_{L,R} \; . \quad (5.37)$$

If we define the **conjugation** of a Weyl spinor by $\chi_L^c = \sigma_2\chi_L^*$ (and similarly for χ_R) then, from Eq. (5.33), the conjugated spinor transforms as

$$\chi_L^c \mapsto \sigma_2\left(R_L(M)\chi_L\right)^* = \sigma_2 R_L(M)^*\chi_L^* \stackrel{(5.33)}{=} R_R(M)\sigma_2\chi_L^* = R_R(M)\chi_L^c \; . \quad (5.38)$$

Hence, the conjugation of a left-handed Weyl spinor is a right-handed Weyl spinor (and vice versa).

How do we construct Lorentz invariants from Weyl spinors. A quadratic expression in two left-handed Weyl spinors transforms in the representation $(1/2,0)\otimes(1/2,0)=(0,0)\oplus(1,0)$ which contains the singlet $(0,0)$. Hence, there must be a corresponding invariant. Eq. (5.34) suggests its form is $\chi_L^T\sigma_2\psi_L$. We have

$$\begin{aligned}\chi_L^T\sigma_2\psi_L \mapsto (R_L(M)\chi_L)^T\sigma_2(R_L(M)\psi_L) &= \chi_L^T R_L(M)^T\sigma_2 R_L(M)\psi_L\\ &= \chi_L^T\sigma_2 R_L(M)^{-1}R_L(M)\psi_L\\ &= \chi_L^T\sigma_2\psi_L\ ,\end{aligned}$$

so this term and its right-handed counterpart

$$\chi_L^T\sigma_2\psi_L\ ,\qquad \chi_R^T\sigma_2\psi_R\ ,\tag{5.39}$$

called **Weyl mass terms**, are indeed Lorentz-invariants. Alternatively, considering that

$$\chi_L^T\sigma_2=-(\sigma_2\chi_L)^T=(\sigma_2\chi_L^*)^\dagger=(\chi_L^c)^\dagger$$

and that χ_L^c is, in fact, a right-handed Weyl spinor, a Weyl mass term can be re-written in the form $(\chi_L^c)^\dagger\psi_L$ or, if right-handed Weyl spinor $\chi_{L.R}$ and $\psi_{R,L}$ are available, as

$$\chi_R^\dagger\psi_L\ ,\qquad \chi_L^\dagger\psi_R\ .\tag{5.40}$$

The tensor of a left and right-handed Weyl spinor, $(1/2,0)\otimes(0,1/2)=(1/2,1/2)$, is a vector so we should be able to make this more explicit. Eq. (5.35) suggests the right way forward.

$$\chi_R^\dagger\sigma_\mu\psi_R\mapsto(R_R(M)\chi_R)^\dagger\sigma_\mu(R_R(M)\psi_R)=\chi_R^\dagger(R_R(M)^\dagger\sigma_\mu R_R(M))\psi_R\overset{(5.35)}{=}\Lambda_\mu{}^\nu\,\chi_R^\dagger\sigma_\nu\psi_R$$

This (and a similar calculation for the left-handed case) shows that

$$\chi_L^\dagger\bar\sigma_\mu\psi_L\ ,\qquad \chi_R^\dagger\sigma_\mu\psi_R\tag{5.41}$$

transform as Lorentz vectors, so we can obtain invariants by contracting these into another Lorentz vector. This facilitates the construction of Lorentz-invariant kinetic terms of the form

$$\chi_L^\dagger\bar\sigma_\mu\partial^\mu\psi_L\ ,\qquad \chi_R^\dagger\sigma_\mu\partial^\mu\psi_R\ .\tag{5.42}$$

For vector fields A_μ, which transform as $A_\mu\mapsto\Lambda_\mu{}^\nu A_\nu$, kinetic terms should be bilinears in $\partial_\mu A_\nu$ but gauge invariance (see later) dictates that we should only consider Lagrangians which depend on the **field strength**, that is, on the anti-symmetric combination $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ with transformation $F_{\mu\nu}\mapsto\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma F_{\rho\sigma}$. With this restriction, there are only two possible Lorentz-invariant terms, namely

$$F_{\mu\nu}F^{\mu\nu}\ ,\qquad \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\ .\tag{5.43}$$

The first of these is the standard kinetic term and it is invariant under L. Note that the second term is only invariant under $\mathrm{SO}(3,1)=L_+^\uparrow\cup L_+^\downarrow$ since it transforms as $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\mapsto\det(\Lambda)\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and, hence, changes sign under parity.

What about terms of higher order than quadratic in the fields, that is, the **coupling terms**? Couplings among scalars are already contained in the Lagrangian density (5.36). To couple scalars to spinors, we can use any of the above bi-linear spinor invariants and multiply them with any function of the scalars. In particular, terms of the form

$$\phi\,\chi_L^T\sigma_2\psi_L\ ,\qquad \phi\,\chi_R^T\sigma_2\psi_R\ ,\tag{5.44}$$

are Lorentz-invariant and are referred to as **Yukawa couplings**. Lorentz-invariant couplings of scalars with vectors are straightforwardly written down, by just contracting indices, for example, schematically

$$A^\mu \phi \partial_\mu \phi \ , \qquad A^\mu A^\nu \partial_\mu \phi \partial_\nu \phi \ . \tag{5.45}$$

Finally, in order to couple vectors to fermions we can use the fermion bilinears (5.41) which transform as vectors and this leads to expressions of the form

$$A^\mu \chi_L^\dagger \bar{\sigma}_\mu \psi_L \ , \qquad A^\mu \chi_R^\dagger \sigma_\mu \psi_R \tag{5.46}$$

Everything so far has been formulated in terms of Weyl spinors but it is easy to combine the above Lorentz invariants into Lorentz invariants for Dirac spinors which are of the form

$$\psi_D = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \tag{5.47}$$

To do this explicitly, it is useful to introduce the **Dirac matrices**

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \quad \Rightarrow \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \ . \tag{5.48}$$

In this context, it is also useful and customary to introduce the conjugation of Dirac spinors defined by

$$\bar{\psi}_D = \psi_D^\dagger \gamma_0 \ . \tag{5.49}$$

Exercise 5.1. *Write down Lorentz invariant terms for Dirac spinors, using gamma matrices. Examine under which conditions combinations of Lorentz-invariant terms of (left- and right-handed) Weyl spinors can be written in terms of Dirac spinors.*

Recall that for a left-handed Weyl spinor χ_L, the conjugate χ_L^c is a right-handed Weyl spinor. This means we can construct a Dirac spinor

$$\chi_M = \begin{pmatrix} \chi_L \\ \chi_L^c \end{pmatrix} \tag{5.50}$$

which contains only one Weyl-spinor, rather than two. Such a spinor is called a **Majorana spinor** and can be used as an alternative (and somewhat redundant) way to describe Weyl spinors. All the above Weyl terms can be re-written in terms of Majorana spinors but we will refrain from doing so.

Yet another way to describe Weyl spinors in terms of Dirac spinors is to project to the left- or right handed part. To this end we note that with

$$\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{5.51}$$

we can define the projection operators $P_L = \frac{1}{2}(1 - \gamma_5)$ and $P_R = \frac{1}{2}(1 + \gamma_5)$ which act on a Dirac spinor (5.49) as

$$\psi_{D,L} := P_L \psi_D = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \ , \qquad \psi_{D,R} := P_R \psi_D = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} \tag{5.52}$$

that is, they project onto the left- and right-handed Weyl-spinors within χ_D. In this way, expressions in terms of Weyl-spinors χ_L and χ_R can be written with left- and right-handed

Diract spinors $\psi_{D,L}$ and $\psi_{D,R}$.

You may be interested in field theories with a different underlying symmetry such as, for example, just rotational symmetry. The rules for building field theories for other symmetries follow similar lines so the above discussion for the Lorentz group can be used as a template.

5.2 The Poincaré group

You may wonder why our discussion of Lorentz-invariant model building did not mention terms with explicit coordinate dependence, such as $x^\mu\partial_\mu\phi$, where ϕ is a scalar field. Such terms, provided indices are contracted properly, are Lorentz invariant. However, they are not invariant under translations $x^\mu \mapsto x^\mu + A^\mu$, another symmetry apparently preserved by fundamental laws of physics. Lorentz transformations together with translations form a group called the **Poincaré group** which is really the full group of space-time symmetries normally assumed for model building.

Definition of Poincaré group

The Poincaré group $\mathcal{P}$ consists of all pairs of Lorentz transformations and translations, so

$$\mathcal{P} = \{(\Lambda, A)\,|\,\Lambda \in L,\ A \in \mathbb{R}^4\}\ , \tag{5.53}$$

and it acts on $\mathbb{R}^4$ as

$$\mathbb{R}^4 \ni x \mapsto \Lambda x + A\ . \tag{5.54}$$

A short calculation for $(\Lambda, A), (\lambda, a) \in \mathcal{P}$ based on this action

$$(\Lambda, A)((\lambda, a)x) = (\Lambda, A)(\lambda x + a) = \Lambda\lambda x + \Lambda a + A = (\Lambda\lambda, \Lambda a + A)x$$

leads to the group multiplication law

$$(\Lambda, A)(\lambda, a) = (\Lambda\lambda, \Lambda a + A) \qquad \Rightarrow \qquad e = (\mathbb{1}_4, 0)\ , \quad (\Lambda, A)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}A) \tag{5.55}$$

for the Poincaré group.

Lie algebra of the Poincaré group

As it stands the Poincaré group is not a matrix Lie group - it consists of pairs of matrices and vectors. One way to calculate the Lie algebra in this case is to go back to our general formalism and work out the left-invariant vector fields on $\mathcal{P}$. This can, in fact, be done. Alternatively, we can define an injective map $p : \mathcal{P} \to \mathrm{GL}(\mathbb{R}^5)$ by

$$p((\Lambda, A)) = \begin{pmatrix} \Lambda & A \\ 0 & 1 \end{pmatrix}\ . \tag{5.56}$$

and note that due to

$$\begin{aligned} p((\Lambda, A))P((\lambda, a)) &= \begin{pmatrix} \Lambda & A \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda\lambda & \Lambda a + A \\ 0 & 1 \end{pmatrix} = p((\Lambda\lambda, \Lambda a + A)) \\ &= p((\Lambda A)(\lambda, a)) \end{aligned}$$

this is actually a group homomorphism which defines an embedding of $\mathcal{P}$ into $\mathrm{GL}(\mathbb{R}^5)$. Hence, we can identity the Poincaré group with the set of matrices (5.56) and use our standard techniques for matrix Lie groups to determine the Lie algebra.

Parametrising the Lorentz group with six parameters, as before, and the translations by themselves the generators of the Poincaré group are

$$J_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} , \qquad P_\mu = \begin{pmatrix} 0 & e_\mu \\ 0 & 0 \end{pmatrix} , \tag{5.57}$$

where e_μ are the four-dimensional standard unit vectors and $\sigma_{\mu\nu}$ are the Lorentz group generators (5.9). Hence, the Lie algebra of the Poincaré group is $\mathcal{L}(\mathcal{P}) = \mathrm{Span}(J_{\mu\nu}, P_\mu)$ with dimension $\dim(\mathcal{P}) = 10$. The commutation relations can be obtained by straightforward matrix calculations (and the $M_{\mu\nu}$ of course commute exactly like the $\sigma_{\mu\nu}$) and this leads to [1]

$$\begin{array}{rcl} [J_{\alpha\beta}, J_{\gamma\delta}] &=& i(\eta_{\alpha\delta}J_{\beta\gamma} + \eta_{\alpha\gamma}J_{\delta\beta} + \eta_{\beta\delta}J_{\gamma\alpha} + \eta_{\beta\gamma}J_{\alpha\delta}) \\ {[J_{\mu\nu}, P_\rho]} &=& i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) = (\sigma_{\mu\nu})_\rho{}^\sigma P_\sigma \\ {[P_\mu, P_\nu]} &=& 0 \end{array} \tag{5.58}$$

These relations show that the **Poincaré algebra** can be written as a semi-direct sum $\mathcal{L}(\mathcal{P}) = \mathcal{T} \oplus_S \mathcal{L}(L)$ of the translations $\mathcal{T} = \mathrm{Span}(P_\mu)$ which form the radical and the Lorentz group algebra $\mathcal{L}(L) = \mathrm{Span}(J_{\mu\nu})$ which is semi-simple, in accordance with Eq. (4.1). It is worth noting that the Poincaré commutation relations (5.58) are also realised by the operators

$$\hat{P}_\mu = -i\partial_\mu , \qquad \hat{J}_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\nu) , \tag{5.59}$$

which are frequently used in the physics literature.

Casimir operators

The **Pauli-Lubanski vector** is defined by

$$W_\mu = -\frac{1}{2}\epsilon_\mu{}^{\nu\rho\sigma} J_{\nu\rho} P_\sigma \tag{5.60}$$

and it turns out that its square $W^2 = W_\mu W^\mu$, as well as $P^2 = P_\mu P^\mu$ are Casimir operators, in the sense that they commute with the entire Poincaré algebra. This can be verified by straightforward calculation using the commutation relations (5.58)

Exercise 5.2. *Verify that W^2 and P^2 commute with all P_μ and $J_{\mu\nu}$.*

This means P^2 and W^2 assume certain values (times the unit matrix) on irreps, which can be used to characterise the representation.

Representations: mass and momentum

We are interested in unitary representations of the Poincaré group and for a group element (Λ, a) we denote the representation matrix by $R(\Lambda, a)$ which acts on a representation vector space V. (We will not proceed with the same mathematical rigour here - it would be too much effort and take too long - but the argument outlined can be cast in a rigorous form.)

[1] To stay in line with physics conventions we have here performed the rescaling $J_{\mu\nu} \mapsto -iJ_{\mu\nu}$ and $P_\mu \mapsto -iP_\mu$ in order to end up with hermitian (rather than anti-hermitian) quantities.

We consider vectors $|m,p\rangle$ which are common eigenvectors under P_μ (and, hence, under P^2) so that

$$P_\mu|m,p\rangle = p_\mu|m,p\rangle\ , \qquad P^2|m,p\rangle = -m^2|m,p\rangle \tag{5.61}$$

with eigenvalues p_μ and $-m^2 = p_\mu p^\mu$. Physically, p_μ is identified with the four-momentum and m with the mass. We would like to work out what happens to such eigenvectors of P_μ under the action of the other generators, $J_{\mu\nu}$.

$$\begin{aligned} P_\rho(\mathbb{1}+i\epsilon^{\mu\nu}J_{\mu\nu})|m,p\rangle &= (P_\rho + i\epsilon^{\mu\nu}J_{\mu\nu}P_\rho + i\epsilon^{\mu\nu}[J_{\mu\nu},P_\rho])\,|m,p\rangle \\ &\overset{(5.58)}{=} (p_\rho + i\epsilon^{\mu\nu}J_{\mu\nu}p_\rho + i\epsilon^{\mu\nu}(\sigma_{\mu\nu})_\rho{}^\sigma p_\sigma)\,|m,p\rangle \\ &= (\delta^\sigma_\rho + i\epsilon^{\mu\nu}(\sigma_{\mu\nu})_\rho{}^\sigma)p_\sigma(\mathbb{1}+i\epsilon^{\mu\nu}J_{\mu\nu})\,|m,p\rangle + \mathcal{O}(\epsilon^2) \end{aligned}$$

Exponentiating this result shows that

$$R(\Lambda,0)\,|m,p\rangle \quad \text{is a state of type} \quad |m,\Lambda p\rangle \quad \text{with momentum} \quad \Lambda p\ . \tag{5.62}$$

Massive representations

If $m^2 = -p_\mu p^\mu \neq 0$ there exists a Lorentz transformation Λ with $\Lambda p = (m,0)$ and the corresponding states $|m,(m,0)\rangle$ are, from Eq. (5.62), rotated into states with the same momentum $(m,0)$ by $R(Q,0)$, where Q is a rotation (contained in the rotation subgroup of the Lorentz group). Hence, the states $|m,(m,0)\rangle$ form a representation of $SO(3)$ and, demanding that this representation be irreducible, finite and have spin j we can write

$$R(Q,0)|m,(m,0),j,\mu\rangle = \sum_{\mu'} R^{(j)}(Q)_{\mu'\mu}\,|m,(m,0),j,\mu'\rangle\ , \tag{5.63}$$

where $R^{(j)}(Q)$ are the spin j representation matrices. The group O(3) in this context, as a subgroup of the Lorentz group which leaves vectors $(m,0)$ invariant, is also referred to as **little group**. A quick calculation shows that

$$\left.\begin{array}{rcl} W_0|m,(m,0),j,\mu\rangle &=& 0 \\ W_i|m,(m,0),j,\mu\rangle &=& mJ_i\,|m,(m,0),j,\mu\rangle \end{array}\right\} \Rightarrow\ W^2\,|m,(m,0),j,\mu\rangle = m^2 j(j+1)\,|m,(m,0),j,\mu\rangle\ ,$$

so, while P^2 gives the total mass (squared) of the representation, W^2 pulls out the total spin.

To complete the story, we still have to consider states with more general momenta p which we define as $|m,p,j,\mu\rangle = R(B(p),0)\,|m,(m,0),j,\mu\rangle$, where $B(p)$ is a boost with $B(p)(m,0) = p$. The claim is that these states, for all p with $p^2 = -m^2$ and all $\mu = -j,\dots,+j$ span the desired representation characterised by mass m and spin j. To verify this we have to show that these states close under Poincaré transformation. We begin with Lorentz transformations.

$$\begin{aligned} R(\Lambda,0)\,|m,p,j,\mu\rangle &= R(\Lambda,0)R(B(p),0)\,|m,(m,0),j,\mu\rangle \\ &\overset{p'=\Lambda p}{=} R(B(p'),0)R(B(p'),0)^{-1}R(\Lambda,0)R(B(p),0)\,|m,(m,0),j,\mu\rangle \\ &= R(B(p'),0)R(\underbrace{B(p')^{-1}\Lambda B(p)}_{\text{rotation}},0)\,|m,(m,0),j,\mu\rangle \\ &= \sum_{\mu'} R^{(j)}(\underbrace{B(\Lambda p)^{-1}\Lambda B(p)}_{\text{Wigner rotation}})_{\mu'\mu}\,|m,\Lambda p,j,\mu'\rangle \end{aligned}$$

Further, by exponentiating $P_\mu |m,p,j,\mu\rangle = p_\mu |m,p,j,\mu\rangle$ we can derive the effect of translations. In summary we have

$$\begin{array}{rcl} R(\Lambda,0)\,|m,p,j,\mu\rangle & = & \sum_{\mu'} R^{(j)}(B(\Lambda p)^{-1}\Lambda B(p))_{\mu'\mu}\,|m,\Lambda p,j,\mu'\rangle \\ R(0,a)\,|m,p,j,\mu\rangle & = & e^{ia\cdot p}\,|m,p,j,\mu\rangle \end{array} \tag{5.64}$$

In summary, the massive representation of the Poincaré group are labelled by the mass $m \neq 0$ and by a spin $j \in \mathbb{Z}/2$ and they are spanned by the states $|m,p,j,\mu\rangle$, for all p with $p^2 = -m^2$ and $\mu = -j,\ldots,+j$ and with the Poincaré group acting as in Eqs. (5.64). Physically, these corresponds to massive states with mass m, four-momentum p satisfying the mass on-shell condition and with total spin j and spin "orientation" $\mu = -j,\ldots,+j$. In particular, this means a massive particle has $2j+1$ spin states.

Massless representations

For representations with zero mass, $m = 0$, the discussion proceeds along similar lines, however, momenta p with $p^2 = 0$ cannot be transformed into the "rest frame" but instead can always be Lorentz transformed to a four-vector of the form $(1,1,0,0)$. This leads to a different little group - the subgroup of the Lorentz group which leaves the vector $(1,1,0,0)$ invariant - which is the Euclidean group E(2) in two dimensions, consisting of two-dimensional orthogonal maps and two-dimensional translations. In addition to momenta p with $p^2 = 0$ massless representations are then determined by representation of this little group, which are labelled by a spin $j \in \mathbb{Z}/2$ and consist of two "helicity states" with $\mu = \pm j$. We refrain from going into the details. The interested reader can, for example, look at Ref. [11], vol. 1, Section 2.5.

5.3 Unitary groups and tensor methods

We now discuss our first class of Lie groups - the unitary groups. After setting up the basics, we will determine their Lie algebras and develop the entire formalism related to the introduction of the Cartan-Weyl basis for these cases.

General properties

Definition of unitary groups

If we introduce the metric

$$\eta = \mathrm{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q}) \tag{5.65}$$

with signature (p,q) and in $n = p+q$ dimensions we can define the unitary group leaving those metrics invariant by

$$\mathrm{U}(p,q) = \{U \in \mathrm{GL}(\mathbb{C}^n)\,|\,\underbrace{U^\dagger \eta U = \eta}_{\Rightarrow|\det(U)|=1}\}$$
$$\mathrm{SU}(p,q) = \{U \in \mathrm{U}(p,q)\,|\,\det(U) = 1\}$$

The more familiar standard unitary groups are the special cases for Euclidean signature $(p,q) = (n,0)$ given by

$$\mathrm{U}(n) = \mathrm{U}(n,0) = \{U \in \mathrm{GL}(\mathbb{C}^n)\,|\,U^\dagger U = \mathbb{1}_n\}$$
$$\mathrm{SU}(n) = \mathrm{SU}(n,0) = \{U \in \mathrm{U}(n)\,|\,\det(U) = 1\}$$

As usual, the n-dimensional representation defined by these matrices is referred to as the fundamental representation.

Application 5.3: *(Unitary groups in physics)*
The groups $\mathrm{SU}(n)$ make a frequent appearance in physics, as groups underlying gauge theories (see later) but also sometimes as global symmetries, for example as in the SU(3) quark model (see later). The groups $(\mathrm{S})\mathrm{U}(p,q)$ for the non-Euclidean cases are not as common but, as so many other mathematical features, appear in string theory, in the context of string duality symmetries.

All group $(\mathrm{S})\mathrm{U}(p,q)$ are path-connected, the groups $(\mathrm{S})\mathrm{U}(n)$ are compact (since $U^\dagger U = \mathbb{1}_n$ means every row of U is normalised to 1 under the standard complex scalar product) but the groups $(\mathrm{S})\mathrm{U}(p,q)$ with $p,q \neq 0$ are non-compact (due to the presence of positive and negative signs in η which means there are no bounds on the matrix entries).

Relation between $\mathrm{SU}(n)$ and $\mathrm{U}(n)$

Define the map $f : \mathrm{U}(1) \times \mathrm{SU}(n) \to \mathrm{U}(n)$ by $f(z,U) = zU$, where $z \in U(1)$ and $U \in \mathrm{U}(n)$. It follows that $\mathrm{Im}(f) = \mathrm{U}(n)$, since a $U \in \mathrm{U}(n)$ is the image of $(\zeta, \zeta^{-1}U)$, where ζ is any solution of $\zeta^n = \det(U)$. On the other hand, if $(z,U) \in \mathrm{Ker}(f)$ it follows that $zU = \mathbb{1}_n$ so that $\det(zU) = z^n = 1$. Hence, z is any of the n^{th} roots of unity and $U = z^{-1}\mathbb{1}_n$, so we learn that $\mathrm{Ker}(f) \cong \mathbb{Z}_n$. In summary, from the isomorphism theorem we have

$$\mathrm{U}(n) \cong \frac{\mathrm{SU}(n) \times U(1)}{\mathbb{Z}_n} \; . \tag{5.66}$$

Lie algebras

Proceeding in the usual fashion by writing $U = 1 + T + \cdots$ and working out the linearised constraint on T amounts to the same calculation we carried out for SU(2). The constraint $U^\dagger \eta U = \eta$ implies that $T = -\eta T^\dagger \eta$ and $\det(U) = 1$ leads to $\mathrm{tr}(T) = 0$, so the Lie algebra is

$$\mathrm{su}(p,q) = \mathcal{L}(\mathrm{SU}(p,q)) = \{T \in \mathrm{End}(\mathbb{C}^n)\,|\, T = -\eta T^\dagger \eta,\ \mathrm{tr}(T) = 0\} \; . \tag{5.67}$$

These matrices are traceless and anti-hermitian in the $(++)$ and $(--)$ entries, relative to the metric η and hermitian in the $(+-)$ and $(-+)$ entries. The dimension only depends on n and can be counted as follows.

$$\dim_{\mathbb{R}}(\mathrm{su}(p,q)) = \underbrace{\frac{n(n-1)}{2}}_{\text{above diagonal}}\, 2 + \underbrace{(n-1)}_{\text{diagonal}} = n^2 - 1 \; . \tag{5.68}$$

For much of our discussion we will use the complexification of this algebra. Since allowing factors of i washes out the difference between hermitian and anti-hermitian matrices all complexifications $\mathrm{su}(p,q)_{\mathbb{C}}$ for a fixed $n = p + q$ are, in fact, the same and are given by

$$\mathrm{su}(p,q)_{\mathbb{C}} = \mathrm{su}(n)_{\mathbb{C}} = \{T \in \mathrm{End}(\mathbb{C}^n)\,|\,\mathrm{tr}(T) = 0\} = \mathrm{sl}(n,\mathbb{C}) =: A_{n-1} \; . \tag{5.69}$$

The standard name for these algebras is A_{n-1} (labelled by their rank, hence the one taken off) and their dimension (with respect to the complex field) is unchanged, $\dim_{\mathbb{C}}(A_{n-1}) = n^2 - 1$.

Cartan-Weyl formalism

Cartan decomposition of A_{n-1}

For convenience of notation, we introduce the standard unit vector e_i, where $i = 1, \ldots, n$ on $\mathbb{C}^n$, the matrices $H_i = \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 appears in the i^{th} position on the diagonal, and the matrices E_{ij} with a 1 in entry (ij) and zero everywhere else. We also define a basis (L_i) dual to the basis (H_i), so that we have the relations $L_i(H_j) = \delta_{ij}$. With this notation the Cartan subalgebra of A_{n-1} can be written as

$$\mathcal{H} = \left\{ \sum_i b_i H_i \,|\, b_i \in \mathbb{C},\ \sum_i b_i = 0 \right\} \qquad \Rightarrow \qquad \mathrm{rk}(A_{n-1}) = n - 1\ . \tag{5.70}$$

The sum constraint of the coefficients b_i is of course the zero trace condition. The dual Cartan $\mathcal{H}'$ is spanned by the L_i but we have to divide out multiples of $L_1 + L_2 + \cdots + L_n$ which acts as the zero functional on $\mathcal{H}$ due to the zero trace constraint.

$$\mathcal{H}' = \frac{\mathbb{C}(L_1, \ldots, L_n)}{\mathbb{C}(L_1 + \cdots + L_n)}\ . \tag{5.71}$$

(Here, $\mathbb{C}(L_1, \ldots, L_n)$ just means the span of all vectors in the list with coefficients in $\mathbb{C}$.) The claim is now that

$$A_{n-1} = \mathcal{H} \oplus \bigoplus_{i \neq j} \mathbb{C}(E_{ij}) \tag{5.72}$$

is the correct Cartan-Weyl decomposition for A_{n-1}. To check this we first note that A_{n-1} is indeed spanned by all the traceless diagonal matrices in $\mathcal{H}$ and the E_{ij} for $i \neq j$. To check the defining eigenvalue property of the Cartan-Weyl decomposition start with $H = \sum_i b_i H_i \in \mathcal{H}$ and work out

$$\mathrm{ad}(H)(E_{ij}) = [H, E_{ij}] = (b_i - b_j)E_{ij} = (\underbrace{L_i - L_j}_{=:L_{ij}})(H)E_{ij}\ .$$

This shows that the E_{ij} are indeed eigenvectors with associated roots $L_{ij} = L_i - L_j$. Hence, the roots space for A_{n-1} is

$$\Delta = \{L_{ij} = L_i - L_j \,|\, i \neq j\}\ . \tag{5.73}$$

Killing form

Since we know the root space we can now work out the Killing form on the Cartan, using the result from Theorem 4.6 (vi). With $H = \sum_i b_i H_i$ and $\tilde{H} = \sum_j \tilde{b}_j H_j$ (where $\sum_i b_i = \sum_i \tilde{b}_i = 0$) this gives

$$\Gamma(H, \tilde{H}) = \sum_{i \neq j} L_{ij}(H) L_{ij}(\tilde{H}) = \sum_{i \neq j} (b_i - b_j)(\tilde{b}_i - \tilde{b}_j) = \sum_{i,j} (b_i - b_j)(\tilde{b}_i - \tilde{b}_j) = 2n \sum_i b_i \tilde{b}_i$$

To find the Killing form on $\mathcal{H}'$ we start with $L = \sum_i l_i L_i \in \mathcal{H}'$ and find the element $H_L = \sum_i c_i H_i \in \mathcal{H}$, which corresponds to L under the isomorphism in Theorem 4.6 (i).

$$\Gamma(H_L, H) = L(H) \quad \forall H \in \mathcal{H} \quad \Leftrightarrow \quad \sum_i (2nc_i - l_i) b_i = 0 \quad \forall b_i \text{ with } \sum_i b_i = 0\ .$$

This implies that $2nc_i - l_i$ must be constant, independent of i and this constant should be fixed so that $\sum_i c_i = 0$, as is appropriate for an element of $\mathcal{H}$. This gives

$$L = \sum_i l_i L_i \;, \quad H_L = \sum_i c_i H_i \qquad \Leftrightarrow \qquad c_i = \frac{1}{2n}(l_i - k) \;, \quad k = \frac{1}{n}\sum_j l_j \;. \tag{5.74}$$

From Theorem 4.6 (i), the Killing form on $\mathcal{H}'$ is defined in terms of the Killing form on $\mathcal{H}$ via this isomorphism, so for $L = \sum_i l_i L_i$ and $\tilde{L} = \sum_j \tilde{l}_j L_j$ we have

$$(L, \tilde{L}) = \Gamma(H_L, H_{\tilde{L}}) = \frac{1}{2n}\sum_i (l_i - k)(\tilde{l}_i - \tilde{k}) = \frac{1}{2n}\left(\sum_i l_i \tilde{l}_i - \frac{1}{n}\sum_i l_i \sum_j \tilde{l}_j\right) \tag{5.75}$$

Positive and negative roots

To distinguish positive from negative roots we need to choose a direction $\ell = \sum_i \ell_i H_i \in \mathcal{H}$ (with $\sum_i \ell_i = 0$) and all we have to demand is that $\ell_1 > \ell_2 > \cdots > \ell_n$. Since $\ell\left(\sum_i l_i L_i\right) = \sum_i \ell_i l_i$ it follows that the positive and negative roots are

$$\Delta_+ = \{L_{ij} \,|\, i < j\} \;, \qquad \Delta_- = \{L_{ij} \,|\, i > j\} \;. \tag{5.76}$$

Further, the positive simple roots are

$$\alpha_i = L_{i,i+1} \quad \text{where} \quad i = 1, \ldots, n-1 \;. \tag{5.77}$$

It is not hard to see that all positive roots can be obtained as sums of the α_i, for example $L_{13} = L_1 - L_3 = (L_1 - L_2) + (L_2 - L_3) = \alpha_1 + \alpha_2$.

Cartan matrix

We should work out the Dynkin label for a weight $\lambda = \sum_i \lambda_i L_i$, from Eq. (5.75). Since the simple positive root α_i corresponds to a vector $(l_i) = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$ we know from Eq. (5.75) that $(\alpha_j, \alpha_j) = 1/n$ and $(\lambda, \alpha_j) = (\lambda_j - \lambda_{j+1})/(2n)$. Combining these results we find for the Dynkin label of λ

$$a_j = \frac{2(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} = \lambda_j - \lambda_{j+1} \in \mathbb{Z} \;. \tag{5.78}$$

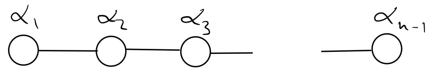
The Cartan matrix is obtained by evaluating this for $\lambda = \alpha_i$ which gives

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\alpha_i)_j - (\alpha_i)_{j+1} = \left\{\begin{array}{rl} 2 & \text{for } i = j \\ -1 & \text{for } |i-j| = 1 \\ 0 & \text{otherwise} \end{array}\right. \;. \tag{5.79}$$

or, as a matrix,

$$A(A_{n-1}) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \;. \tag{5.80}$$

Cartan matrices can also be represented graphically, by a **Dynkin diagram**. This is a diagram with $\mathrm{rk}(\mathcal{L})$ nodes, where node i and node j are connected by $-A_{ij}$ links for $i < j$. Following this rule the Dynkin diagram of A_{n-1} is



Weights

Eq. (5.78) shows that $\lambda = \sum_i \lambda_i L_i$ is in the weight lattice iff all differences $\lambda_i - \lambda_{i+1}$ are integer. Using the quotient in Eq. (5.71) we can always choose a representative such that all λ_i are integers, simply by subtracting the non-integer part times $L_1 + \ldots + L_n$. This still leaves us to divide out integer multiples of $L_1 + \cdots + L_n$ so that the weight lattice is

$$\Lambda_W = \frac{\mathbb{Z}(L_1, \ldots, L_n)}{\mathbb{Z}(L_1 + \cdots + L_n)} \ . \tag{5.81}$$

The remaining quotient can be used to choose a representative with $\lambda_n = 0$ and then Eqs. (5.78) can be solved for λ_i in terms of the Dynkin label a_i. The result is that a Dynkin label $(a_1, \ldots, a_{n-1})$ corresponds to a weight

$$\lambda = \sum_i \lambda_i L_i = (a_1 + \cdots + a_{n-1})L_1 + (a_2 + \cdots + a_{n-1})L_2 + \cdots + a_{n-1}L_{n-1} \ . \tag{5.82}$$

Representations

We know that representations are classified by highest weight Dynkin labels $(a_1, \ldots, a_{n-1})$ with all $a_i \geq 0$. But how does such a seemingly abstract statement actually help? As we will see, it is actually quite powerful when combined with tensor methods which allow us to construct the irreps of A_{n-1}.

We start modestly by looking at the fundamental representation on $V \cong \mathbb{C}^n$. Since $E_{ij}e_1 = 0$ for all $i < j$ we learn that e_1 is the highest weight vector of the fundamental. (To see this recall the definition of the highest vector from Def. 4.8 and from Eq. (5.76) that E_{ij} for $i < j$ are the raising operators). Further, we have

$$He_i = \left(\sum_j b_j H_i \right) e_i = b_i e_i = L_i(H) e_i \ ,$$

which shows that e_i is a vector with weight L_i. In conclusion, we see that the fundamental is a representation with highest weight L_1 which, from Eq. (5.82), corresponds to a highest weight Dynkin label $(1, 0, \ldots, 0)$.

This does not seem like much progress yet but let's take symmetric and anti-symmetric tensor powers. First note that if a vector space U carries a representation, then so do the anti-symmetric rank k tensors $\Lambda^k U$ and the symmetric rank k tensors $S^k U$ (every index is acted on by the same representation matrix so symmetrisation and anti-symmetrisation are preserved under the action of the representation).

Consider the anti-symmetric power $\Lambda^k V$ of the fundamental. Since $E_{ij}e_l = \delta_{jl}e_i$ for $i < j$ its highest weight is $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ with weight $L_1 + L_2 + \cdots + L_k$ (recall that e_i has weight L_i and that weights add under tensoring) which, from Eq. (5.82) translates into the highest weight Dynkin label $(0, \ldots, 0, 1, 0, \ldots 0)$ with the 1 in the k^{th} position. So in short, the anti-symmetric powers $\Lambda^k V$ of the fundamental for $k = 1, \ldots, n-1$ provide us with the "unit vector" Dynkin labels.

Consider a representation on a vector space U with highest weight vector $u \in U$ and highest weight λ. Then, the representation on $S^k U$ has highest weight vector u^k with highest weight $k\lambda$.

Combining these statements we see that the A_{n-1} irrep with highest weight Dynkin label $(a_1, \ldots, a_{n-1})$ is contained in the tensor product

$$S^{a_1}(V) \otimes S^{a_2}(\Lambda^2 V) \otimes \cdots \otimes S^{a_{n-1}}(\Lambda^{n-1} V) \tag{5.83}$$

of the fundamental V and it has highest weight vector

$$v = e_1^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes \cdots \otimes (e_1 \wedge \cdots e_{n-1})^{a_{n-1}} \ . \tag{5.84}$$

So with a few short arguments we have shown that all A_{n-1} representations can be obtained as tensors of the fundamental, suitably symmetrised and anti-symmetrised. The space (5.83) is not always irreducible but it certainly contains the irrep with highest weight Dynkin label $(a_1, \ldots, a_{n-1})$. We will soon discuss more sophisticated methods to symmetrise which lead to irreps but for now it is useful to include some discussion on how to calculate with tensors explicitly.

Tensors, hands-on

Basic set-up

Consider the fundamental representation $\mathbf{n}$ and the complex conjugate of the fundamental [2] $\bar{\mathbf{n}}$ of $\mathrm{SU}(n)$. If we denote elements in the underlying representation vector spaces by ϕ_μ for $\mathbf{n}$ and ϕ^μ for $\bar{\mathbf{n}}$, where $\mu, \nu = 1, \ldots n$, then their $\mathrm{SU}(n)$ transformation is

$$\phi_\mu \mapsto U_\mu{}^\nu \phi_\nu \ , \qquad \phi^\mu \mapsto \bar{U}^\mu{}_\nu \phi^\nu \tag{5.85}$$

where $U \in \mathrm{SU}(n)$ with entries $U_\mu{}^\nu$ and $\bar{U}$ is the complex conjugate of U with entries $\bar{U}^\mu{}_\nu$. A (p,q) tensor in $\mathbf{n}^p \otimes \bar{\mathbf{n}}^q$ carries p lower and q upper indices, $\phi^{\nu_1 \cdots \nu_q}_{\mu_1 \cdots \mu_p}$, and transforms as

$$\phi^{\nu_1 \cdots \nu_q}_{\mu_1 \cdots \mu_p} \mapsto U_{\mu_1}{}^{\rho_1} \cdots U_{\mu_p}{}^{\rho_p} \, \bar{U}^{\nu_1}{}_{\sigma_1} \cdots \bar{U}^{\nu_q}{}_{\sigma_q} \, \phi^{\sigma_1 \cdots \sigma_q}_{\rho_1 \cdots \rho_p} \tag{5.86}$$

Symmetrisation and anti-symmetrisation in any number of lower or upper indices is preserved under this action since all lower and upper indices are acted upon by the same matrix.

Exercise 5.3. *Let $\phi_{\mu\nu}$ be an $\mathrm{SU}(n)$ tensor with $\phi_{\mu\nu} = \pm\phi_{\mu\nu}$. Show explicitly that the (anti-)symmetry of the tensor is preserved under the action of $\mathrm{SU}(n)$.*

Another useful ingredient is that the Kronecker delta δ_μ^ν and the Levi-Civita tensors $\epsilon_{\mu_1 \cdots \mu_n}$ and $\epsilon^{\mu_1 \cdots \mu_n}$ are $\mathrm{SU}(n)$ invariant, as a direct result of the defining relations $U^\dagger U = \mathbb{1}_n$ and $\det(U) = 1$ of $\mathrm{SU}(n)$. For this reason, they can be used to construct new tensors from given ones. The point is that the transformation property of a tensor - and the corresponding $\mathrm{SU}(n)$ representation - can be read off from the index structure.

[2] It is common in physics to denote representations by their dimensions (in boldface, say). This is not the greatest notation as representations are certainly not always uniquely characterised by their dimensions. But it is fairly widely used and physicists are ruthless that way, so here we go.

Examples of SU(n) tensors

(1) For a $(1,1)$ tensor ϕ_μ^ν the contraction $\phi_\mu^\mu = \delta_\nu^\mu \phi_\mu^\nu$ is an invariant.

(2) For an $(n,0)$ tensor $\phi_{\mu_1\cdots\mu_n}$ and a $(0,n)$ tensor $\phi^{\mu_1\cdots\mu_n}$ the contractions $\epsilon^{\mu_1\cdots\mu_n}\phi_{\mu_1\cdots\mu_n}$ and $\epsilon_{\mu_1\cdots\mu_n}\phi^{\mu_1\cdots\mu_n}$ are invariants.

(3) For an $(n-1,0)$ tensor $\phi_{\mu_1\cdots\mu_{n-1}}$ we can define a $(0,1)$ tensor by $\phi^\mu = \epsilon^{\mu\mu_1\cdots\mu_{n-1}}\phi_{\mu_1\cdots\mu_{n-1}}$, which, hence, transforms as an $\bar{\mathbf{n}}$ representation. We have seen that the fundamental, $\mathbf{n}$, realised on a vector space $V \cong \mathbb{C}^n$, has a highest weight Dynkin label $(1,0,\ldots,0)$. Now it is clear that the complex conjugate of the fundamental, $\bar{\mathbf{n}}$ is realised on $\wedge^{n-1}V$. This fits with the tensor classification in Eq. (5.83) and shows that the highest weight Dynkin label of $\bar{\mathbf{n}}$ is $(0,\ldots,0,1)$.

(4) Consider a $(1,1)$ tensor ϕ_μ^ν. Written as a matrix it transforms as $\phi \mapsto U\phi U^\dagger$, so like the adjoint. But its dimension is n^2 while the dimension of the adjoint is n^2-1. This is an example for a representation on one of the tensor spaces in Eq. (5.83), in this case on $V\otimes(\wedge^{n-1}V)$, which leads to a reducible representation. We can extract the irreducible pieces by writing

$$\phi_\mu^\nu = \underbrace{\left(\phi_\mu^\nu - \frac{1}{n}\delta_\mu^\nu\phi_\rho^\rho\right)}_{\text{adjoint}} + \frac{1}{n}\delta_\mu^\nu \underbrace{\phi_\rho^\rho}_{\text{singlet}} \tag{5.87}$$

This gives us a Clebsch-Gordan decomposition $\mathbf{n}\otimes\bar{\mathbf{n}} = \mathbf{1}\oplus\mathbf{adj}$ and also shows that the highest weight Dynkin label of the adjoint is $(1,0,\ldots,0,1)$.

(5) Consider SU(3) and a $(2,0)$ tensor $\phi_{\mu\nu} = \phi_{(\mu\nu)} + \phi_{[\mu\nu]}$, where the round (square) brackets around indices denote symmetrisation (anti-symmetrisation). The symmetric tensor $\phi_{(\mu\nu)}$ is six-dimensional and denoted by $\mathbf{6}$. It lives in S^2V and, hence, has Dynkin label $(2,0)$. The anti-symmetric part $\phi^\mu = \epsilon^{\mu\nu\rho}\phi_{[\nu\rho]}$ is a $\bar{\mathbf{3}}$ complex conjugate fundamental, so we have the Clebsch-Gordan decomposition $\mathbf{3}\otimes\mathbf{3} = \bar{\mathbf{3}}\oplus\mathbf{6}$.

(6) Repeating this discussion with SU(5), the $(2,0)$ tensor $\phi_{\mu\nu} = \phi_{(\mu\nu)} + \phi_{[\mu\nu]}$ now has a symmetric piece $\phi_{(\mu\nu)}$ in S^2V of dimension $\mathbf{15}$ and Dynkin label $(2,0,0,0)$ and an anti-symmetric piece $\phi_{[\mu\nu]}$ in $\wedge^2V$ of dimension $\mathbf{10}$ and with Dynkin label $(0,1,0,0)$. Hence, we have the Clebsch-Gordon decomposition $\mathbf{5}\otimes\mathbf{5} = \mathbf{10}\oplus\mathbf{15}$. There is also the complex conjugate fundamental $\bar{\mathbf{5}}$ with tensor ϕ^μ and Dynkin label $(0,0,0,1)$.

Constructing SU(n) invariants

The tensor notation makes it quite simple to write down the SU(n) invariants which can be constructed from given tensors. For example, consider SU(5) with tensors $\phi_{[\mu\nu]}$, $\tilde{\phi}_{[\mu\nu]}$ in $\mathbf{10}$, tensor $\bar{H}^\mu$ and ψ^μ in $\bar{\mathbf{5}}$ and a tensor H_μ in $\mathbf{5}$. Then, obvious SU(5) invariants are

$$\epsilon^{\mu\nu\rho\sigma\tau}\phi_{[\mu\nu]}\tilde{\phi}_{[\rho\sigma]}H_\tau\,, \qquad \phi_{[\mu\nu]}\psi^\mu\bar{H}^\nu\,, \qquad \tilde{\phi}_{[\mu\nu]}\psi^\mu\bar{H}^\nu\,. \tag{5.88}$$

Branchings for SU(n)

It is easy to embed smaller unitary groups into larger ones and then work our branchings of representations. This is best illustrated with an example, say $\mathrm{SU}(2)\times\mathrm{SU}(3)\subset\mathrm{SU}(5)$, with the explicit embedding

$$\mathrm{SU}(2)\times\mathrm{SU}(3)\ni(U_3,U_2)\mapsto\begin{pmatrix}U_3 & 0\\ 0 & U_2\end{pmatrix}\in\mathrm{SU}(5)\ .$$

Let $\mu,\nu=1,\dots,5$ denote SU(5) indices, $a,b,\dots=1,2,3$ are SU(3) indices and $i,j,\dots=4,5$ are SU(2) indices. Hence, indices are "broken up" as $\mu=(a,i)$. Branchings can then be easily worked out by specialising the SU(5) indices on a tensor to all possible combinations of SU(2) and SU(3) indices and reading off the resulting representation.

$$\begin{array}{rcl} \mathbf{5}\ni\phi_\mu & \mapsto & (\phi_a,\phi_i)\\ \bar{\mathbf{5}}\ni\phi^\mu & \mapsto & (\phi^a,\phi^i)\\ \mathbf{10}\ni\phi_{[\mu\nu]} & \mapsto & (\phi_{[ab]},\phi_{ai},\phi_{[ij]})\end{array}\qquad\begin{array}{rcl}\mathbf{5} & \mapsto & (\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})\\ \bar{\mathbf{5}} & \mapsto & (\bar{\mathbf{3}},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})\\ \mathbf{10} & \mapsto & (\bar{\mathbf{3}},\mathbf{1})\oplus(\mathbf{3},\mathbf{2})\oplus(\mathbf{1},\mathbf{1})\end{array}\tag{5.89}$$

For the second line we note that $\mathbf{2}$ and $\bar{\mathbf{2}}$ of SU(2) are equivalent, so the bar has been dropped. In the last line, $\phi_{[ab]}$ is, in fact, a $\bar{\mathbf{3}}$ of SU(3) from point (3) above. Finally, $\phi_{[ij]}$ is indeed an SU(2) singlet from point (2) above.

Tensors systematically

As we have seen the tensor (5.83) contains the A_{n-1} irrep with highest-weight Dynkin label $(a_1,\dots,a_{n-1})$ but it is unfortunately not always irreducible, as the decomposition (5.87) of the $(1,1)$ tensor shows. To get to irreducible tensors we need a more sophisticated method to symmetrise/anti-symmetrise than provided by Eq. (5.83).

Tensors and Young tableaux

To do this we start with a vector space $V\cong\mathbb{C}^n$ carrying the fundamental representation and rank d tensors in $V^{\otimes d}$ carrying the induced tensor representation. We can represent the permutation group S_d on $V^{\otimes d}$, basically by the elements of S_d permuting the d tensor indices or, more formally, by $\sigma(v_1\otimes\cdots\otimes v_d)=v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(d)}$. This action of S_d commutes with the action of A_{n-1} since every vector in the tensor product is transformed with the same representation matrix (or, in index notation, every index transforms with the same matrix). We can now recycle some of the formalism we have seen in the context of representations for permutation groups. Consider a partition $\lambda=(\lambda_1,\dots,\lambda_{n-1})$ of d, that is, $d=\lambda_1+\cdots+\lambda_{n-1}$ with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{n-1}\geq 0$. To such a partition we can assign a Young tableau with at most $n-1$ rows, and each row with length λ_i, for example, for $d=9$ and $\lambda=(4,2,2,1)$, we have the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4\\ \hline 5 & 6 & \multicolumn{2}{c}{}\\ \cline{1-2} 7 & 8 & \multicolumn{2}{c}{}\\ \cline{1-2} 9 & \multicolumn{3}{c}{}\\ \cline{1-1}\end{array}\ .$$

We can then define the map $P_\lambda:V^{\otimes d}\to V^{\otimes d}$ by

$$P_\lambda=c\left[\sum_{\sigma\in R_\lambda}\sigma\right]\left[\sum_{\sigma\in C_\lambda}\mathrm{sgn}(\sigma)\sigma\right]\tag{5.90}$$

where R_λ and C_λ are the sets of permutations which leave the rows and columns of the Young tableau λ invariant. This map is a projector (for a suitable choice of the number c) and it commutes with the action of A_{n-1} (since every permutation does) and hence

$$P_\lambda(V):=P_\lambda V^{\otimes d}\tag{5.91}$$

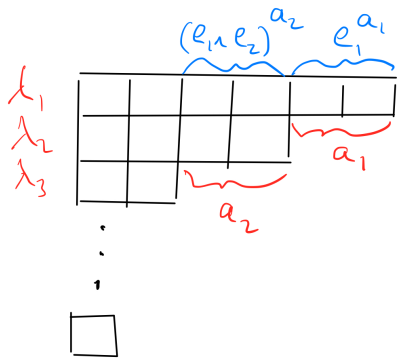
is an A_{n-1} representation. Note that this amounts to symmetrising the tensor in the indices that correspond to each row and anti-symmetrising in the indices corresponding to each column.

Theorem 5.4. *The representation $P_\lambda(V)$ defined above has the following properties.*
(i) It contains the irrep with highest weight $\lambda = \sum_i \lambda_i L_i$ and highest weight Dynkin label $a_i = \lambda_i - \lambda_{i+1}$.
(ii) It is irreducible.

Proof. (i) It is enough to show that $P_\lambda(V)$ contains the highest weight vector

$$v = e_1^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{n-1})^{a_{n-1}}$$

of the irrep with highest weight Dynkin label $(a_1, \ldots, a_{n-1})$. This amounts to showing that $P_\lambda v = v$ and the reason this is true is indicated in the figure below.



Note that the length differences of the Young tableau rows are precisely the highest weight Dynkin labels of the irrep. It is therefore easy to convert between the Young tableau and Dynkin label picture.
(ii) See Ref. [1], p. 223. □

Corollary 5.1. *The Young tableaux with less than n rows are in one-to-one correspondence with the irreps of A_{n-1}.*

Dimension of representations

We can find the dimension of an irrep with Young tableau λ by counting the independent components of the associated vector. This can be done by filling in the numbers $1, \ldots n$ into the boxes of the Young tableau in a way that respects the symmetries enforced by P_λ, that is, symmetries along the rows and anti-symmetry along the columns.

Definition 5.1. *A standard tableau for a Young tableau λ is the Young tableau with the numbers $1, \ldots, n$ filled into the boxes such that*
(i) numbers do not decrease from left to right in each row (symmetry in the rows)
(ii) numbers increase from top to bottom in each column (anti-symmetry in the columns)

Then the dimension of a Young tableau λ equals the number of its standard tableaux, so

$$\dim(P_\lambda(V)) = \#\text{standard tableaux for } \lambda \, . \tag{5.92}$$

Examples

It is useful to practice this with a few examples, to start with for A_{n-1} for generic n.

representation	Young tableau	tensor	standard tableaux	dimension
$\mathbf{n}$	$\boxed{m}$	ϕ_m	$\boxed{1},\ \boxed{2}, \ldots, \boxed{n}$	n
$(\mathbf{n}\otimes\mathbf{n})_{\text{symm}}$	$\begin{array}{\|c\|c\|}\hline m & n \\ \hline\end{array}$	$\phi_{(mn)}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array},\ \begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}, \ldots$	$n(n+1)/2$
$(\mathbf{n}\otimes\mathbf{n})_{\text{anti-symm}}$	$\begin{array}{\|c\|}\hline m \\ \hline n \\ \hline\end{array}$	$\phi_{[mn]}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array},\ \begin{array}{\|c\|}\hline 1 \\ \hline 3 \\ \hline\end{array}, \ldots$	$n(n-1)/2$

Some more specific example for the case SU(3) are:

representation	Young tableau	tensor	Dynkin label
$\mathbf{1}$	$\begin{array}{\|c\|}\hline m \\ \hline n \\ \hline p \\ \hline\end{array}$	$\phi_{[mnp]} = \epsilon_{mnp}\phi$	$(0,0)$
$\mathbf{3}$	$\boxed{m}$	ϕ_m	$(1,0)$
$\bar{\mathbf{3}}$	$\begin{array}{\|c\|}\hline m \\ \hline n \\ \hline\end{array}$	$\phi_{[mn]}$	$(0,1)$
$\mathbf{6}$	$\begin{array}{\|c\|c\|}\hline m & n \\ \hline\end{array}$	$\phi_{(mn)}$	$(2,0)$
$\bar{\mathbf{6}}$	$\begin{array}{\|c\|c\|}\hline m & n \\ \hline p & q \\ \hline\end{array}$	$\phi_{([mp][nq])} = \epsilon_{mpr}\epsilon_{nqs}\phi^{rs}$	$(0,2)$
$\mathbf{8}$	$\begin{array}{\|c\|c\|}\hline m & n \\ \hline p \\ \cline{1-1}\end{array}$	$\star$	$(1,1)$

Let us illustrate how to get the tensor for $\mathbf{8}$ from projecting with $P_{(2,1)}$ we label the three boxes of the Young tableau as

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \Rightarrow \qquad R_{(2,1)} = \left\{ e, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \right\} , \quad C_{(2,1)} = \left\{ e, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right\} .$$

From Eq. (5.90) the projector for this representation is then

$$P_{(2,1)} = \left(e + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \right) \left(e - \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \right) = e + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Applying this to (the indices of) the tensor $\phi_{\mu_1\mu_2\mu_3}$ gives

$$P_\lambda \phi_{\mu_1\mu_2\mu_3} = \phi_{\mu_1\mu_2\mu_3} + \phi_{\mu_2\mu_1\mu_3} - \phi_{\mu_3\mu_2\mu_1} - \phi_{\mu_3\mu_1\mu_2} .$$

To illustrate the counting of dimensions, we write down all eight standard tableaux for $\lambda = (2,1)$, which are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} .$$

Clebsch-Gordan decomposition

Now that we have described representation in terms of Young tableaux we should be able to formulate tensor products and their Clebsch-Gordan decompositions in this language, so write

$$P_\lambda(V) \otimes P_\mu(V) = \bigoplus_v N_{\lambda\mu\nu} P_\nu(V) \ , \tag{5.93}$$

where the integers $N_{\lambda\mu\nu}$ describe how many times the irrep $P_\nu(V)$ appears in the tensor product of the irreps $P_\lambda(V)$ and $P_\mu(V)$. There is a practical algorithm for how to extract these numbers called the Littlewood-Richardson rule (for a proof see Ref. [1], App. A) which proceeds as follows.

Algorithm *(Clebsch-Gordan decomposition from Young tableaux)* To tensor A_{n-1} representations for two Young tableaux λ and μ proceed as follows:

(1) Write the first Young tableau λ as

$$\begin{array}{|c|c|c|c|c|c|}\hline a&a&a&a&a&a\\\hline b&b&b&\multicolumn{3}{c}{}\\\cline{1-3} c&c&\multicolumn{4}{c}{}\\\cline{1-2}\end{array} \quad .$$

(2) Attach boxes of λ to μ, starting with a's, then b's etc, such that (i) no two same letters appear in the same column (ii) the result is always a Young tableaux.

(3) For each Young tableaux obtained in (2), read all letters from right to left and top to bottom. This sequence must form a lattice permutation, that is, to the left of any symbol there are no fewer a' than b's, etc. Otherwise the tableaux is discarded.

At least for low-dimensional examples this algorithm leads to the Clebsch-Gordon decomposition quickly, as the following SU(3) examples show.

$$\begin{array}{ccccccc}
\begin{array}{|c|}\hline a\\\hline\end{array} & \otimes & \begin{array}{|c|}\hline \\\hline\end{array} & = & \begin{array}{|c|}\hline \\\hline a\\\hline\end{array} & \oplus & \begin{array}{|c|c|}\hline &a\\\hline\end{array}\\
\mathbf{3} & \otimes & \mathbf{3} & = & \bar{\mathbf{3}} & \oplus & \mathbf{6}\\
\begin{array}{|c|}\hline a\\\hline\end{array} & \otimes & \begin{array}{|c|}\hline \\\hline \\\hline\end{array} & = & \begin{array}{|c|}\hline \\\hline \\\hline a\\\hline\end{array} & \oplus & \begin{array}{|c|c|}\hline &a\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\\
\mathbf{3} & \otimes & \bar{\mathbf{3}} & = & \mathbf{1} & \oplus & \mathbf{8}\\
\begin{array}{|c|}\hline a\\\hline\end{array} & \otimes & \begin{array}{|c|c|}\hline &\\\hline\end{array} & = & \begin{array}{|c|c|}\hline &\\\hline a&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} & \oplus & \begin{array}{|c|c|c|}\hline &&a\\\hline\end{array}\\
\mathbf{3} & \otimes & \mathbf{6} & = & \mathbf{8} & \oplus & \mathbf{10}
\end{array} \tag{5.94}$$

SU(3) in detail

The Gell-Mann matrices

The group SU(3) is quite commonly used in physics, in the context of the quark model for hadrons and also as a gauge theory, and it makes sense to look at it in more detail. There is an analogue of the Pauli matrices for the SU(3) case, and these are called **Gell-Mann matrices**.

$$\begin{array}{rclrclr}
\lambda_i &=& \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} & i &=& 1,2,3 & \mathrm{su}_I(2) \\
\lambda_4 &=& \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &=& \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \mathrm{su}_V(2) \\
\lambda_6 &=& \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda_7 &=& \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \mathrm{su}_U(2) \\
\lambda_8 &=& \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} & & & &
\end{array} \tag{5.95}$$

The Lie-algebra su(3) is eight-dimensional and it consists of hermitian [3], traceless 3×3 matrices and the above matrices clearly form a basis of this space. The first second and third line correspond to the (non-diagonal) generators which span the su(2) algebras which originate from the three different ways of embedding SU(2) into SU(3) (which are also conventionally labelled by I (for isospin), V and U),

$$\begin{array}{rcl}
\mathrm{SU}_I(2) \ni U & \mapsto & \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SU}(3) \\
\mathrm{SU}_V(2) \ni U & = & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ \gamma & 0 & \delta \end{pmatrix} \in \mathrm{SU}(3) \\
\mathrm{SU}_U(2) \ni U & \mapsto & \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \in \mathrm{SU}(3)
\end{array} \tag{5.96}$$

The Gell-Mann matrices satisfy the nice normalisation property

$$\mathrm{tr}(\lambda_I \lambda_J) = 2\delta_{IJ} \ . \tag{5.97}$$

The generators are taken to be $T_I = \lambda_I/2$, so that $\mathrm{su}(3) = \mathrm{Span}(T_I)$ and group elements are obtained by exponentiating,

$$U = \exp(it^I T_I) \in \mathrm{SU}(3) \ , \tag{5.98}$$

where $t^T \in \mathbb{R}$ and the factor of i appears since we are working in the "physics convention" where the generators are taken to be hermitian (rather than anti-hermitian).

Cartan-Weyl basis

The rank of su(3) is two, so the Cartan has a two-dimensional basis, $\mathcal{H} = \mathrm{Span}(T_3, Y)$ which are basically the two diagonal Gell-Mann matrices suitably normalised

$$T_3 = \mathrm{diag}(1/2, -1/2, 0) \ , \qquad Y = \frac{2}{\sqrt{3}} T_8 = \mathrm{diag}(1/3, 1/3, -2/3) \ . \tag{5.99}$$

Another element in the Cartan which is worth defining is

$$Q = T_3 + \frac{1}{2} Y = \mathrm{diag}(2/3, -1/3, -1/3) \in \mathcal{H} \ . \tag{5.100}$$

[3] For the purpose of this part we stick to the physics convention by writing $U = \mathbb{1}_n + iT + \cdots$, so that the generators T are hermitian.

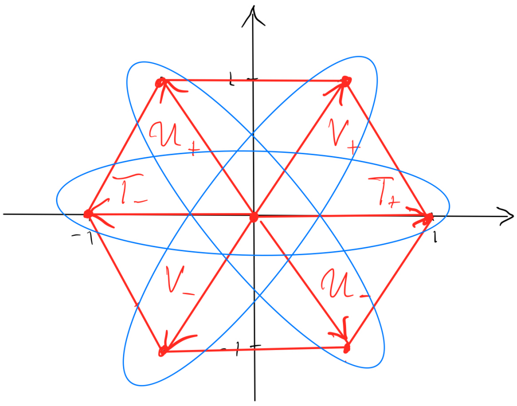
In the context of the SU(3) quark model, Q is identified with electrical charge. To find the raising and lowering operators we can consider the three su(2) sub-algebras and define the standard su(2) raising and lowering operator for each.

$$T_\pm = T_1 \pm iT_2 \ , \qquad V_\pm = T_4 \pm iT_5 \ , \qquad U_\pm = T_6 \pm iT_7 \ . \tag{5.101}$$

Relating to the previous general notation for the Cartan-Weyl basis, the basis (H_i) for $\mathcal{H}$ is here chosen to be (T_3, Y) and the raising and lowering operators (E_α) are $(T_\pm, U_\pm, V_\pm)$. To check that this is indeed the correct Cartan-Weyl basis we should check that the eigenvalue equations $[H_i, E_\alpha] = \alpha_i E_\alpha$ are satisfied - this will also allow us to read off the eigenvalues (the components of the roots) α_i. A straightforward matrix computation gives

$$\begin{array}{rclrclcrcl} [T_3, T_\pm] &=& \pm T_\pm & [Y, T_\pm] &=& 0 & \Rightarrow & \alpha_{T_\pm} &=& (\pm 1, 0) \\ [T_3, U_\pm] &=& \mp\frac{1}{2}U_\pm & [Y, U_\pm] &=& \pm U_\pm & \Rightarrow & \alpha_{U_\pm} &=& (\mp 1/2, \pm 1) \\ [T_3, V_\pm] &=& \pm\frac{1}{2}V_\pm & [Y, V_\pm] &=& \pm V_\pm & \Rightarrow & \alpha_{V_\pm} &=& (\pm 1/2, \pm 1) \end{array} \tag{5.102}$$

The root system can be visualised by a root diagram, a simple drawing of all the roots vectors.



It contains three root diagrams of su(2), indicated by the blue ellipses, which correspond to the three subalgebras $\mathrm{su}_I(2)$, $\mathrm{su}_V(2)$ and $\mathrm{su}_U(2)$.

Fundamental representation and its complex conjugate

The fundamental representation acts on $V \cong \mathbb{C}^3$ and we note the standard unit vector basis by $u = e_1$, $d = e_2$ and $s = e_3$. With the two Cartan generators in the $\mathbf{3}$ fundamental irrep

$$T_3^{(\mathbf{3})} = \mathrm{diag}(1/2, -1/2, 0) \ , \quad Y^{(\mathbf{3})} = \mathrm{diag}(1/3, 1/3, -2/3) \ , \quad Q^{(\mathbf{3})} = \mathrm{diag}(2/3, -1/3, -1/3) \ .$$

it is easy to work out the weights of u, d, s (which are of course common eigenvectors of $T_3^{(\mathbf{3})}$ and $Y^{(\mathbf{3})}$ as

$$\begin{array}{rclrclcrcl} T_3^{(\mathbf{3})}u &=& \frac{1}{2}u & Y^{(\mathbf{3})}u &=& \frac{1}{3}u & \Rightarrow & \lambda_u &=& (1/2, 1/3) \\ T_3^{(\mathbf{3})}d &=& -\frac{1}{2}d & Y^{(\mathbf{3})}d &=& \frac{1}{3}d & \Rightarrow & \lambda_d &=& (-1/2, 1/3) \\ T_3^{(\mathbf{3})}s &=& 0 & Y^{(\mathbf{3})}s &=& -\frac{2}{3}s & \Rightarrow & \lambda_s &=& (0, -2/3) \end{array} \tag{5.103}$$

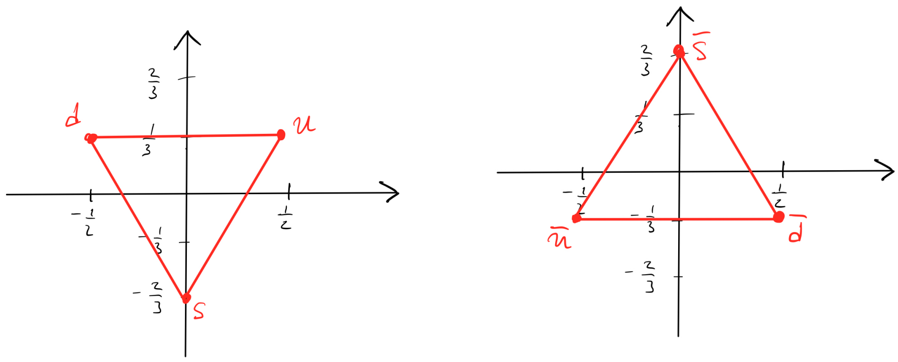
For the complex conjugate of the fundamental, $\bar{\mathbf{3}}$, we introduce the basis $\bar{u} = e_1$, $\bar{d} = e_2$ and $\bar{s} = e_3$. From Eq. (5.98) the generators in $\bar{\mathbf{3}}$ are obtained from those in $\mathbf{3}$ by $T \mapsto -T^*$, so in particular

$$T_3^{(\bar{\mathbf{3}})} = \mathrm{diag}(-1/2, 1/2, 0) \ , \quad Y^{(\bar{\mathbf{3}})} = \mathrm{diag}(-1/3, -1/3, 2/3) \ , \quad Q^{(\bar{\mathbf{3}})} = \mathrm{diag}(-2/3, 1/3, 1/3) \ .$$

Hence, the weights of the $\bar{\mathbf{3}}$ representation are

$$\begin{array}{rclrclcrcl} T_3^{(\bar{\mathbf{3}})}\bar{u} &=& -\frac{1}{2}\bar{u} & Y^{(\bar{\mathbf{3}})}\bar{u} &=& -\frac{1}{3}\bar{u} & \Rightarrow & \lambda_{\bar{u}} &=& (-1/2,-1/3) \\ T_3^{(\bar{\mathbf{3}})}\bar{d} &=& \frac{1}{2}\bar{d} & Y^{(\bar{\mathbf{3}})}\bar{d} &=& -\frac{1}{3}\bar{d} & \Rightarrow & \lambda_{\bar{d}} &=& (1/2,-1/3) \\ T_3^{(\bar{\mathbf{3}})}\bar{s} &=& 0 & Y^{(\bar{\mathbf{3}})}\bar{s} &=& \frac{2}{3}\bar{s} & \Rightarrow & \lambda_{\bar{s}} &=& (0,2/3) \end{array} , \tag{5.104}$$

that is, the negative of the weights in the fundamental, in line with our general statement about the weights of the dual representation. The weights of a representation can be visualised by a **weight diagram** a simple drawing of the weight vectors, as in the figure below.



5.4 Applications

Application 5.4: *(Quark model of mesons)*

There are six quarks, $q = (u, d, s, c, b, t)^T$ (up, down, strange, charm, bottom, top) and we could decide that they furnish a fundamental representation of SU(6), so transform as $q \mapsto Uq$, where $U \in \mathrm{SU}(6)$. In fact, much of the QCD Lagrangian is invariant under this "flavour symmetry", with the exception of the mass Lagrangian

$$\mathcal{L}_{\mathrm{mass}} = \sum_{q=u,d,s,c,b,t} m_q\, \bar{q}q$$

If all m_q were equal this Lagrangian would be SU(6) invariant but this is of course not the case. However, it is true that $m_u, m_d, m_s \ll \Lambda_{\mathrm{QCD}}$, where Λ_{QCD} is the characteristic energy scale of the strong interactions. For this reason, SU(3), with the quarks (u, d, s) forming a fundamental representation $\mathbf{3}$ (and $(\bar{u}, \bar{d}, \bar{s})$ a complex conjugate fundamental, $\bar{\mathbf{3}}$), is an *approximate* symmetry of QCD. The masses of the remaining three quarks (c, b, t) are not negligible compared to Λ_{QCD}, so SU(3) for the three lightest quarks is the largest flavour symmetry we should consider.

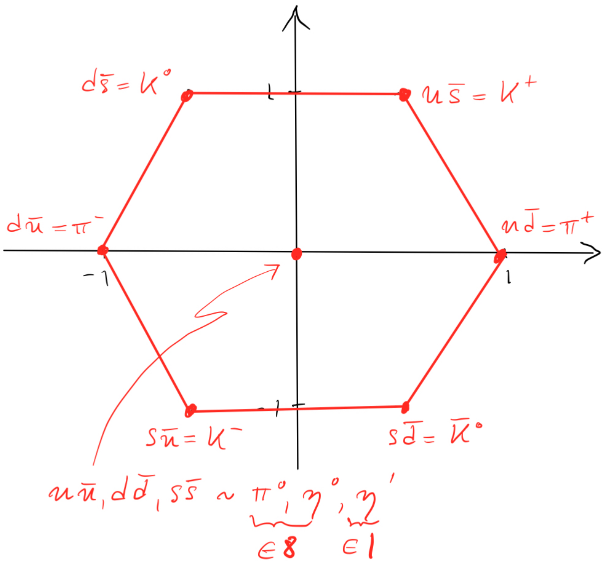
Mesons are quark-antiquark bound states so under SU(3), they should transform as

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \, .$$

We can work out the weights in this tensor product by forming all possible sums of the weights in $\mathbf{3}$ and $\bar{\mathbf{3}}$ (as determined above). This leads to

tensor state	$u\otimes\bar{u}$	$u\otimes\bar{d}$	$u\otimes\bar{s}$	$d\otimes\bar{u}$	$d\otimes\bar{d}$	$d\otimes\bar{s}$	$s\otimes\bar{u}$	$s\otimes\bar{d}$	$s\otimes\bar{s}$
(T_3, Y)	(0,0)	(1,0)	(1/2,1)	(-1,0)	(0,0)	(-1/2,1)	(-1/2,-1)	(1/2,-1)	(0,0)

The resulting weight diagram is shown in the figure below, with the physical names for the (spin 0) mesons included.



The electric charge of these states is given by the Cartan element

$$Q = T_3 + \frac{1}{2}Y\ . \tag{5.105}$$

Of course this diagram consists of the root diagram of SU(3) (since the **8** is the adjoint) plus the weight for the additional singlet at the origin. From the three states $u\otimes\bar{u}$, $d\otimes\bar{d}$ and $s\otimes\bar{s}$ two linear combinations must be part of the octet (and they are called π^0 and η^0) and one must be singlet (which is called η'). How can we identify those linear combinations? To see this write down the following lowering operators

$$\begin{aligned} T_-^{(\mathbf{3})} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & T_-^{(\bar{\mathbf{3}})} &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ U_-^{(\mathbf{3})} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & U_-^{(\bar{\mathbf{3}})} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{5.106}$$

in the **3** and $\bar{\mathbf{3}}$ representation and then construct their counterparts in the $\mathbf{3}\otimes\bar{\mathbf{3}}$ representation given by

$$T_-^{(\mathbf{3}\otimes\bar{\mathbf{3}})} = \mathbb{1}_3 \times T_-^{(\bar{\mathbf{3}})} + T_-^{(\mathbf{3})} \times \mathbb{1}_3\ , \qquad U_-^{(\mathbf{3}\otimes\bar{\mathbf{3}})} = \mathbb{1}_3 \times U_-^{(\bar{\mathbf{3}})} + U_-^{(\mathbf{3})} \times \mathbb{1}_3\ . \tag{5.107}$$

If we apply these operators to states in **8** the result remains in **8**. In particular, applying $T_-^{(\mathbf{3}\otimes\bar{\mathbf{3}})}$ to $u\bar{d}$ and $U_-^{(\mathbf{3}\otimes\bar{\mathbf{3}})}$ to $d\bar{s}$ should lead to two states with weight $(0,0)$ which are both in the octet.

$$T_-^{(\mathbf{3}\otimes\bar{\mathbf{3}})}(u\bar{d}) = u(T_-^{(\bar{\mathbf{3}})}\bar{d}) + (T_-^{(\mathbf{3})}u)\bar{d} = -u\bar{u} + d\bar{d}\ , \qquad U_-^{(\mathbf{3}\otimes\bar{\mathbf{3}})}(d\bar{s}) = -d\bar{d} + s\bar{s}$$

Hence, normalising and forming orthogonal linear combinations, we conclude that

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \in \mathbf{8}\ , \quad \eta^0 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \in \mathbf{8}\ , \quad \eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \in \mathbf{1}\ .$$

Suppose, we would like to identify which isospin $\mathrm{SU}_I(2) \subset \mathrm{SU}(3)$ multiplets a given SU(3) multiplet contains. The branchings

$$\mathbf{3} \mapsto \mathbf{2} \oplus \mathbf{1}\ , \qquad \bar{\mathbf{3}} \mapsto \mathbf{2} \oplus \mathbf{1}\ ,$$

are immediately clear from the embedding (5.96). Just as easily it follows that

$$\mathbf{8} \oplus \mathbf{1} = \mathbf{3} \otimes \bar{\mathbf{3}} \mapsto (\mathbf{2} \oplus \mathbf{1}) \otimes (\mathbf{2} \oplus \mathbf{1}) = \mathbf{3} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}$$

so that the octet branches under $\mathrm{SU}_I(2)$ as

$$\mathbf{8} \mapsto \mathbf{3} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1}\ . \tag{5.108}$$

Hence, the octet contains an isospin triplet (the pions π^-, π^0, π^+), two isospin doublets (K^0, K^+ and $K^-, \bar{K}^0$) and an isospin singlet (η^0).

Baryons are bound states of three quarks so they furnish the representation

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{3} \otimes (\bar{\mathbf{3}} \oplus \mathbf{6}) = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}\ , \tag{5.109}$$

where the Clebsch-Gordan decomposition follows immediately from the Young tableau calculations (5.94). This leads to the baryon octet and decouplet and the details can be worked out along similar lines as for the mesons. More details on the quark model of hadrons can be found in most particle physics textbooks, for example in Ref. [12].

Application 5.5: *(Model building with internal global (Lie) symmetries - another rough guide)*

In Application (5.2) we have considered the constraints imposed by Lorentz symmetry on field theory Lagrangians. In this context, the Lorentz symmetry is also referred to as an **external symmetry** - the symmetry acting on the space-time coordinates. Field theories can also have symmetries which do not act on the space-time coordinates but on the "field space" coordinates and such symmetries are also referred to as **internal symmetries**. In fact, the SU(3) flavour symmetry we have just introduced is an example of such a symmetry (although it is merely an approximate symmetry).

To discuss this more explicitly, consider a field $\phi = (\phi^a)$, where $a = 1, \ldots, n$. For simplicity, we focus on scalar fields here but analogous considerations can be carried out for fermions. At every point in space time the field takes values in a vector space, $\phi(x) \in \mathbb{F}^n$, (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, depending on whether the ϕ^a are real or complex valued). Let us assume that we have a Lie group G with representation $R : G \to \mathrm{GL}(\mathbb{F}^n)$ and that the field ϕ transforms under this representation, so $\phi(x) \mapsto R(g)\phi(x)$. Here we are using the same $R(g)$ for every space-time point x and such a symmetry G is also called a **global symmetry**. (We will discuss local symmetries=gauge symmetries later.) How can we build G-invariant Lagrangians?

In general, we can allow a certain type of term if the Clebsch-Gordan decomposition of the corresponding representations contains a singlet. For example, a certain linear combination of quadratic terms $\phi^a\phi^b$ is allowed if $R\otimes R$ contains a singlet. If R is a unitary representation (this is usually the case in physics applications) then $R^*\otimes R$ contains a singlet which is, in fact, given by $\phi^\dagger\phi$ (since $R(g)^\dagger R(g) = \mathbb{1}_n$). This means the following terms

$$\partial_\mu\phi^\dagger\partial^\mu\phi\ , \qquad \phi^\dagger\phi\ , \qquad (\phi^\dagger\phi)^2\ , \tag{5.110}$$

are all G-invariant. The first of these is a kinetic term, the second a mass term and the third a quartic coupling term. These may well not be all the allowed terms but details depend on the symmetry.

As an example, consider $G = \mathrm{SU}(2)$ with $\phi = (\phi^a)$, where $a = 1,2$, transforming in the fundamental, $\mathbf{2}$. Which terms, besides the ones in Eq. (5.110), are allowed in this case? Since $\mathbf{2}\otimes\mathbf{2} = \mathbf{1}\oplus\mathbf{3}$ a quadratic term must be allowed. From $U^* = \sigma_2 U\sigma_2$ (recall that $\mathbf{2}$ is pseudo-real) and $U^\dagger U = \mathbb{1}_2$ it follows that $U^T\sigma_2 U = \sigma_2$ for $U\in\mathrm{SU}(2)$ and, hence,

$$\phi^T\sigma_2\phi\ , \qquad (\phi^T\sigma_2\phi)^2 \tag{5.111}$$

are invariants. On the other hand, since $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = \mathbf{2}\oplus\mathbf{2}\oplus\mathbf{4}$ contains no singlet there is no invariant cubic in ϕ.

Next consider $G = \mathrm{SU}(3)$ with $\phi = (\phi^a)$, where $a = 1,2,3$, transforming in the fundamental, $\mathbf{3}$. Since $\mathbf{3}\otimes\mathbf{3} = \bar{\mathbf{3}}\oplus\mathbf{6}$ contains no singlet no quadratic term of the form ϕ^2 is allowed. (If we use a complex conjugate we have of course $\phi^\dagger\phi$, as in Eq. (5.110), which matches the fact that $\mathbf{3}\otimes\bar{\mathbf{3}}$ contains a singlet.) However, since $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ contains a singlet, there is an invariant cubic term which is, in fact, given by

$$\epsilon_{abc}\phi_1^a\phi_2^b\phi_3^c\ . \tag{5.112}$$

(Here, we are using three SU(3) triplet fields $\phi_i = (\phi_i^a)$, where $i = 1,2,3$, since the coupling (5.112) vanishes due to anti-symmetry of ϵ_{abc} if only one field is used.)

Exercise 5.5. *Construct invariant terms for other groups and representations.*

Application 5.6: *(Spontaneous breaking of (global) Lie symmetries)*

Symmetries of field theory are typically (partially or completely) broken by specific solutions to the theory. The simplest type of solutions to (scalar) field theories are minima of the scalar potential and in this application we would like to discuss symmetry breaking by such solutions.

Suppose as before that we have a scalar ϕ which takes values in $\phi(x)\in\mathbb{R}^n$ and a Lie group G with representation $R: G\to \mathrm{GL}(\mathbb{R}^n)$ (and Lie-algebra counterpart $r:\mathcal{L}(G)\to\mathrm{End}(\mathbb{R}^n)$) under which ϕ transforms as $\phi\mapsto R(g)\phi$ [4]. Assume a G-invariant Lagrange density

$$\mathcal{L} = \partial_\mu\phi^T\partial^\mu\phi - V(\phi)$$

with a G-invariant scalar potential V, so that

$$V(\phi) = V(R(g)\phi) \tag{5.113}$$

[4] For simplicity we work with real-valued scalar fields. Complex-valued fields can always be split up into two real-valued fields, so our discussion remains general.

for all $g \in G$. If $v = \langle\phi\rangle$ is a minimum of V (that is, there is a neighbourhood of v such that $V(\phi) \geq V(v)$ for all ϕ in this neighbourhood), then so is $R(g)v$ for all $g \in G$. Minima of invariant potentials never come in isolation but in orbits generated by the invariance group! On the set $\mathcal{M}$ of all minima of V (the "moduli space" of V) we can define an equivalence relation by saying that two minima v and $\tilde{v}$ are related if there exists a $g \in G$ and a $\lambda \in \mathbb{R}$ such that $\tilde{v} = \lambda R(g)v$. In this way $\mathcal{M}$ splits into disjoint equivalence classes $\{\lambda R(g)v \,|\, g \in G,\ \lambda \in \mathbb{R}\}$ of orbits under G. For a given $v \in \mathcal{M}$ the unbroken subgroup of G is

$$H_v = \{g \in G \,|\, R(g)v = v\} \qquad \Rightarrow \qquad \mathcal{L}(H_v) = \{T \in \mathcal{L}(G) \,|\, r(T)v = 0\}\ . \tag{5.114}$$

It is easy to show that $H_{\lambda R(g)v} = gH_v g^{-1}$, which means that the unbroken group (up to conjugation) does not depend on v but only on the G-orbit of $\mathcal{M}$ (and we simply call it H from now on). Since Eq. (5.114) implies that v is a singlet under the branching of R under H there is an important group-theoretical constraint on symmetry breaking: For a scalar ϕ in a G-representation R to be able to generate a breaking to a subgroup $H \subset G$ it is a necessary condition that the branching of R under H contains an H-singlet.

Write $\phi = v + \varphi$ and expand the potential around v as

$$V(\phi) = V(v) + \frac{1}{2}M_{ab}\varphi^a\varphi^b + \mathcal{O}(\varphi^3)\ , \qquad M_{ab} = \frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(v)\ , \tag{5.115}$$

where M is called the **mass matrix**. Writing Eq. (5.113) as $V(\phi) = V(R(g(t))\phi)$, where $t = (t^i)$ are the parameters of the group, and differentiating this equation with respect to t^i leads to

$$0 = \frac{\partial V}{\partial\phi_a}(r(T_i)\phi)_a\ .$$

As this holds on a basis T_i of generators it holds for all generators T and taking a derivative with respect to ϕ_b and evaluating at $\phi = v$ gives

$$M(r(T)v) = 0\ . \tag{5.116}$$

If $T \in \mathcal{L}(H)$ then $r(T)v = 0$ and this equation becomes trivial. However, for $T \notin \mathcal{L}(H)$ we have $r(T)v \neq 0$ which is, hence, an eigenvector with eigenvalue zero of the mass matrix M. This means that we have one massless mode for every "broken" generator. This statement is referred to as **Goldstone's theorem** and the massless modes are also called **Goldstone modes**. We conclude that

$$\dim(G) - \dim(H) = \#\ \text{Goldstone modes}\ . \tag{5.117}$$

As an example consider the group $G = \mathrm{SU}(5)$ and the two obvious subgroups $H_1 = \mathrm{SU}(4) \subset \mathrm{SU}(5)$ and $H_2 = \mathrm{SU}(3) \times \mathrm{SU}(2) \subset \mathrm{SU}(5)$. If $R = \mathbf{5}$ we have the branchings

$$\mathbf{5} \mapsto [\mathbf{4} \oplus \mathbf{1}]_{H_1}\ , \qquad \mathbf{5} \mapsto [(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})]_{H_2}\ .$$

From the above discussion this means that a scalar ϕ in the fundamental $\mathbf{5}$ can break to H_1 but not to H_2. A minimum value v transforms as $v \mapsto Uv$, where $U \in \mathrm{SU}(5)$, and can, hence, always be rotated (and scaled) to $v = (0,0,0,0,1)^T$. This means that $\mathcal{M}$ has only a single orbit on which SU(4) is unbroken since it leaves $v = (0,0,0,0,1)^T$ invariant. Since $\dim(\mathrm{SU}(5)) = 24$ and $\dim(\mathrm{SU}(4)) = 15$, there are 9 Goldstone modes.

Consider the same set-up but with $R = \mathbf{24}$, the adjoint of SU(5). In this case ϕ takes values in the Lie algebra of SU(5), so the hermitian (say), traceless 5×5 matrices and transforms as $\phi\mapsto U\phi U^\dagger$. Using standard tensor methods we have the branchings

$$\mathbf{24}\mapsto[\mathbf{15}\oplus\mathbf{4}\oplus\bar{\mathbf{4}}\oplus\mathbf{1}]_{H_1}\;,\qquad \mathbf{24}\mapsto[(\mathbf{8},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{3},\mathbf{2})\oplus(\bar{\mathbf{3}},\mathbf{2})\oplus(\mathbf{1},\mathbf{1})]_{H_2}$$

Since both branchings contain a singlet the adjoint can facilitate breakings which preserve H_1 and H_2. To understand the orbits we can diagonalise to $v=\mathrm{diag}(v_1,\dots,v_5)$, where $v_i\in\mathbb{R}$ and $\sum_i v_i=0$. The values of the four independent v_i (modulo overall scaling) classify the orbits. For generic choices neither of H_1 and H_2 are unbroken and only $U(1)^4$ survives (with 24-4=20 resulting Goldstone modes). For the non-generic choice $v=\mathrm{diag}(v,v,v,v,-4v)$ $H_1\times U(1)$ is unbroken (with $24-15-1=8$ Goldstone modes) and for $\mathbf{v}=\mathrm{diag}(2v,2v,2v,-3v,-3v)$ $H_2\times U(1)$ is unbroken (with $24-8-3-1=12$ Goldstone modes).

Application 5.7: *(Local symmetries and gauge theories - a rough guide)*
Consider a theory with external symmetry group G and fields ϕ transforming as $\phi\mapsto R(g)\phi$ in the (unitary) representation $R:G\to\mathrm{GL}(\mathbb{F}^n)$ of G. The corresponding Lie algebra representation is $r:\mathcal{L}(G)\to\mathrm{End}(\mathbb{F}^n)$. In the case of a global symmetry ϕ transforms in the same way for all space-time points x, so $\phi(x)\to R(g)\phi(x)$ for all x and with $R(g)$ independent of x. A **local symmetry** or **gauge symmetry** allows for transformations of ϕ which depend on x, so we should think of the group elements $g=g(x)$ and their representation matrices $G(x)=R(g(x))$ as functions of x, while ϕ now transforms as $\phi(x)\mapsto G(x)\phi(x)$. Much of what we have said about how to build theories invariant under global symmetries remains valid for local symmetries [5]. In particular, all Lagrangian terms without derivatives which are globally invariant remain locally invariant. However, we have to be careful with terms involving derivatives since their transformation will produce additional contributions proportional to $\partial_\mu G$. More specifically, we have

$$\partial_\mu\phi\mapsto\partial_\mu(G\phi)=G(\partial_\mu\phi+G^{-1}\partial_\mu G\phi)\;. \tag{5.118}$$

The second term is new, compared to the global case, and it prevents, for example, a standard kinetic term $\partial_\mu\phi^\dagger\partial^\mu\phi$ from being invariant under the local symmetry. The additional term can be removed by introducing new fields with the right transformation. Since $G^{-1}\partial_\mu G$ takes values in $\mathcal{L}(G)$ and carries a Lorentz index, the new field $A_\mu=A_\mu^aT_a$ should be a vector field taking values in $\mathcal{L}(G)$, where T_a are the generators of $\mathcal{L}(G)$. This field, called the **gauge field**, together with its counterpart $A_\mu^{(r)}=r(A_\mu)=A_\mu^aT_a^{(r)}$ in the representation r transform as

$$A_\mu\mapsto gA_\mu g^{-1}-\partial_\mu g\,g^{-1}\;,\qquad A_\mu^{(r)}\mapsto GA_\mu^{(r)}G^{-1}-\partial_\mu G\,G^{-1}\;. \tag{5.119}$$

The first term is just the standard transformation in the adjoint representation and the second inhomogeneous term arises from the local nature of the symmetry. It is of course designed to cancel the unwanted term in Eq. (5.118). With the **gauge covariant derivative**

$$D_\mu^{(r)}\phi=\partial_\mu\phi+A_\mu^{(r)}\phi \tag{5.120}$$

[5] We will take a somewhat informal approach to gauge theories here, to keep the mathematical overhead in check. Mathematically, gauge theories are formulated in terms of principle and vector bundles over space-time manifolds. In this context, fields“ transforming” under the gauge symmetry are sections of vector bundles and the gauge field itself is a local manifestation of a bundle connection. More details can, for example, be found in Ref. [13].

a quick calculation shows that

$$D_\mu^{(r)}\phi \mapsto (\partial_\mu + GA_\mu^{(r)}G^{-1} - \partial_\mu G\, G^{-1})(G\phi) = GD_\mu^{(r)}\phi \tag{5.121}$$

so that the modified kinetic term $(D_\mu^{(r)}\phi)^\dagger D^{(r)\mu}\phi$ is gauge invariant. More generally, the replacement $\partial_\mu \mapsto D_\mu^{(r)}$ is the way to convert globally invariant theories into gauge invariant ones. Note that this process introduces (cubic and quartic) interaction terms between ϕ and the gauge field.

In order to introduce kinetic terms for the gauge field we introduce its field strength

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\ , \qquad F_{\mu\nu} \mapsto gF_{\mu\nu}g^{-1}\ . \tag{5.122}$$

A gauge-invariant kinetic term for the gauge field is then

$$-\frac{1}{4g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\ , \tag{5.123}$$

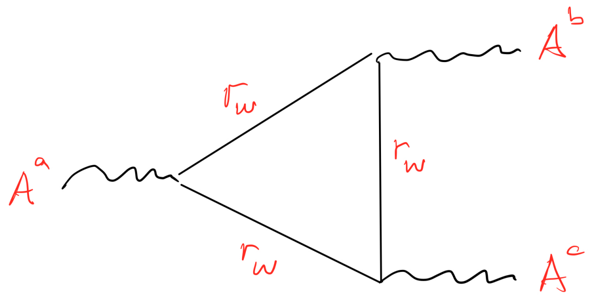
where g is the **gauge coupling constant** [6].

In group-theoretical terms, a gauge theory is characterised by the gauge group G, the representation R_S of scalar fields and the representation R_W of left-handed Weyl fermions [7], with Lie algebra counterparts r_S and r_W. Since the gauge field takes values in the adjoint its associated representation is the adjoint representation, ad, and the number of gauge fields is

$$\#\text{ gauge fields} = \dim(\mathrm{ad}) = \dim(G)\ . \tag{5.124}$$

Of course we can also consider Dirac spinors in a representation r_D. However, if we split the Dirac spinor up as $\phi_D = (\chi_L, \chi_R)$ into left- and right-handed Weyl spinors, it can also be described by the left-handed Weyl spinors χ_L, χ_R^c with representation $r_W = r_D \oplus \bar{r}_D$.

An interesting feature of gauge theories is the possibility of **anomalies**, that is, a break-down of gauge symmetry at the quantum level. It can be shown that such anomalies can be detected by the diagrams



with three external gauge fields A^a, A^b and A^c and the Weyl fermions in r_W in the loop. It turns out [8] these diagrams (setting $r = r_W$) are proportional to

$$A_{abc}^{(r)} = \mathrm{tr}\left(T_a^{(r)}\{T_b^{(r)}, T_c^{(r)}\}\right)\ , \tag{5.125}$$

[6] A field redefinition $A_\mu \mapsto gA_\mu$ moves the gauge coupling into the interaction terms which is the usual convention in physics.

[7] Right-handed Weyl spinors χ_R can always be converted into left-handed ones, χ_R^c, by conjugation.

[8] See the QFT course.

and only if these quantities vanish is the theory anomaly-free. How do the anomalies for r and its complex conjugate $\bar{r}$ relate? We have $T_a^{(\bar{r})} = (T_a^{(r)})^*$, so

$$A_{abc}^{(\bar{r})} = \mathrm{tr}\left(T_a^{(\bar{r})}\{T_b^{(\bar{r})}, T_c^{(\bar{r})}\}\right) = \mathrm{tr}\left((T_a^{(r)})^\dagger\{(T_b^{(r)})^\dagger, (T_c^{(r)})^\dagger\}\right) = -A_{abc}^{(r)} , \tag{5.126}$$

where we use the convention with anti-hermitian generators. On the other hand, if r is real or pseudo-real, so that $T_a^{(\bar{r})} = PT_a^{(r)}P^{-1}$ it follows easily from the properties of the trace that $A_{abc}^{(\bar{r})} = A_{abc}^{(r)}$. This means that a theory with Weyl fermions in a real or pseudo-real representation r_W is anomaly-free. In particular, a theory written in terms of Dirac spinors is anomaly free since the representation r_D translates into the representation $r_W = r_D \oplus \bar{r}_D$ for Weyl fermions which is real.

An interesting special case is a gauge group of the form $\mathrm{U}(1) \times G$ with Weyl spinors in representations r with $U(1)$ charge q and generators $q\mathbb{1}$ for $\mathrm{U}(1)$ and $T_a^{(r)}$ for G. In this case, we have a "mixed anomaly" with one $\mathrm{U}(1)$ and two G gauge fields with anomaly coefficient

$$A_{ab} = 2\sum_{(r,q)} q\,\mathrm{tr}(T_a^{(r)}T_b^{(r)}) = -2\sum_{(r,q)} q\, c(r)\delta_{ab} , \tag{5.127}$$

where $c(r)$ is the index of r. In this case, the anomaly vanishes iff

$$\sum_{(r,q)} q\, c(r) = 0 , \tag{5.128}$$

For a gauge group $\mathrm{U}(1)$ with a set of Weyl fermions with $\mathrm{U}(1)$ charges q_i the theory is anomaly-free iff

$$\sum_i q_i^3 = 0 . \tag{5.129}$$

Another interesting effect for gauge theories is the running of the gauge coupling g with energy μ, due to quantum effects, which is governed by the differential equation

$$\mu\frac{dg}{d\mu} = \beta(g) , \tag{5.130}$$

where $\beta(g)$ is referred to as the **beta function**. It can be computed in perturbation theory and to one-loop level it is given by the formula

$$\beta(g) = -\frac{1}{16\pi^2}\left[\frac{11}{3}c(\mathrm{ad}) - \frac{2}{3}c(r_W) - \frac{1}{6}c(r_S)\right] g^3 , \tag{5.131}$$

where the three terms arise form the contributions of the gauge fields, the (Weyl) fermions and the scalars, respectively. Deriving this formula is of course a task for a QFT course but the point here is that the entire problem (to this order) is determined by group-theoretical quantities, namely the indices of the representations involved.

Application 5.8: *(The standard model of particle physics)*

The standard model of particle physics is a gauge theory with gauge group $G_{\mathrm{SM}} = \mathrm{SU}_c(3) \times \mathrm{SU}_W(2) \times \mathrm{U}_Y(1)$, where the first factor leads to $\dim(\mathrm{SU}(3)) = 8$ gauge bosons, called **gluons** which mediate the strong interaction and the other factors to $\dim(SU(2) \times U(1)) = 4$ gauge bosons which mediate the electro-weak interactions (one combination of which is the photon). A family of standard model (Weyl) fermions is contained in the G_{SM} representation

$$\begin{array}{ccccccccc} (\mathbf{3},\mathbf{2})_1 & \oplus & (\bar{\mathbf{3}},\mathbf{1})_{-4} & \oplus & (\bar{\mathbf{3}},\mathbf{1})_2 & \oplus & (\mathbf{1},\mathbf{2})_{-3} & \oplus & (\mathbf{1},\mathbf{1})_6 \\ Q^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix} & & u_R^{ic} & & d_R^{ic} & & L^i = \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix} & & e_R^{ic} \end{array} \tag{5.132}$$

where the first and second entry correspond to the $SU_c(3)$ and $SU_W(2)$ representations and the subscript is the $U_Y(1)$ charge (normalised so that all charges are integers). The index $i = 1, 2, 3$ labels the three families and the indices in the $\mathbf{3}$ or $\bar{\mathbf{3}}$ representations have been suppressed. The conventional electrical charge is given by

$$Q = \tau_3 + \frac{1}{6}Y \ , \tag{5.133}$$

where $\tau_3 = \sigma_3/2$ is the $su_W(2)$ generator in the fundamental and Y is the $U_Y(1)$ charge, also called the **weak hypercharge**. There is also a scalar multiplet, the **Higgs multiplet** in

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \sim (\mathbf{1},\mathbf{2})_3 \ . \tag{5.134}$$

Spontaneous symmetry breaking $SU_W(2) \times U_Y(1) \to U_Q(1)$ is induced by the vacuum expectation value $\langle H \rangle = (0, v)^T$. It satisfies $Q\langle H \rangle = 0$ and, therefore, does indeed leave the electric charge (5.133) unbroken.

The standard model is anomaly free. For anomalies of the type $SU(3)^3$ this follows since the quarks can be combined into Dirac spinors (u_L^i, u_R^i) and (d_L^i, d_R^i). For anomalies $SU(2)^3$ it follows because the only representations which occur are the $SU(2)$ singlet and the fundamental which is pseudo-real. Anomalies $SU(2)SU(3)^2$ and $SU(3)SU(2)^2$ vanish from Eq. (5.125) since the $SU(2)$ and $SU(3)$ generators are traceless and for the same reason anomalies of type $U(1)^2SU(2)$ and $U(1)^2SU(3)$ vanish. The vanishing of the remaining anomalies is less trivial and depends on the precise $U(1)$ charges and from Eqs. (5.128), (5.129), we have

$$\begin{array}{lcccl} U(1)SU(2)^2 & : & A & \sim & 3Y_Q + Y_L = 0 \\ U(1)SU(3)^2 & : & A & \sim & 2Y_Q + Y_u + Y_d = 2 - 4 + 2 = 0 \\ U(1)^3 & : & A & \sim & 6Y_Q^3 + 3Y_u^3 + 3Y_d^3 + 2Y_L^3 + Y_e^3 \\ & & & = & 6 - 192 + 24 - 54 + 216 = 0 \end{array} \tag{5.135}$$

Most modern quantum field theory books, including Refs. [11, 12, 14], have sections on the standard model. A dedicated book is Ref. [15].

Application 5.9: *(Unification)*
Unification is the attempt of obtaining the standard model from a (spontaneously broken) gauge theory with a gauge group G which contains G_{SM} as a sub-group and whose matter field representations r branch under G_{SM} into the representation (5.132) for a standard model family (or at least a representation that contains (5.132)). The idea is of course to have a simple (or at least semi-simple) G and a matter representation r which looks less baroque than (5.132).

Let us consider a very simple-minded attempt with $G = \mathrm{SU}_c(3) \times \mathrm{SU}_W(3) \supset G_{\mathrm{SM}}$, where we keep the colour group $\mathrm{SU}_c(3)$ unchanged and try to "unify" the electro-weak group into $\mathrm{SU}_W(3) \supset \mathrm{SU}_W(2) \times \mathrm{U}_Y(1)$, using the embedding

$$\mathrm{SU}_W(2) \ni U \mapsto \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SU}_W(3) \;, \qquad \mathrm{U}_Y(1) \ni \alpha \mapsto \mathrm{diag}(\alpha, \alpha, \alpha^{-2}) \in \mathrm{SU}_W(3) \;. \tag{5.136}$$

Such a theory has $\dim(\mathrm{SU}(3) \times \mathrm{SU}(3)) = 16$ gauge bosons, four more than the standard model. Its basic representations branch as

$$(\mathbf{3}, \mathbf{3}) \mapsto (\mathbf{3}, \mathbf{2})_1 \oplus (\mathbf{3}, \mathbf{1})_{-2} \;, \quad (\bar{\mathbf{3}}, \mathbf{1}) \mapsto (\bar{\mathbf{3}}, \mathbf{1})_0 \;, \quad (\mathbf{1}, \mathbf{3}) \mapsto (\mathbf{1}, \mathbf{2})_1 \oplus (\mathbf{1}, \mathbf{1})_{-2}$$

While we can obtain the correct $\mathrm{SU}_c(3) \times \mathrm{SU}_W(2)$ representations this way (although we need three $\mathrm{SU}(3) \times \mathrm{SU}(3)$ irreps for this) the $\mathrm{U}_Y(1)$ charges are clearly not consistent with the ones in Eq. (5.132). So this simple attempt fails for basic group-theoretical reasons.

Exercise 5.6. *Explore other possibilities for unification, for example based on the groups* $\mathrm{SU}(5)$ *or* $\mathrm{SU}(6)$*.*

Basic unification models are described in some quantum field theory books, for example, Refs. [12, 15]. Much of the underlying group theory is described in Ref. [4]. A dedicated book is Ref. [16].

5.5 Orthogonal groups and spinors

General properties

Definition of orthogonal groups

As for the unitary groups we start slightly more general than usual and introduce the metrics

$$\eta = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}) \tag{5.137}$$

with signature (p, q), where $n = p + q$. Then we define

$$\mathrm{O}(p, q) = \{A \in \mathrm{GL}(\mathbb{R}^n) \,|\, A^T \eta A = \eta\} \;, \quad \mathrm{SO}(p, q) = \{A \in \mathrm{O}(p, q) \,|\, \det(A) = 1\} \tag{5.138}$$

as well as the more standard orthogonal group for the Euclidean case, $\eta = \mathbb{1}_n$,

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(\mathbb{R}^n) \,|\, A^T A = \mathbb{1}_n\} \;, \quad \mathrm{SO}(n) = \{A \in \mathrm{O}(n) \,|\, \det(A) = 1\} \;. \tag{5.139}$$

As usual, the n-dimensional representation these matrices define is referred to as fundamental representation. For $A \in \mathrm{O}(p, q)$ it follows that $\det(A) \in \{\pm 1\}$ so these group consists of (at least) two path-disconnected parts, the positive determinant part $\mathrm{SO}(p, q)$ (which is connected) and the part which consists of negative determinant matrices.

Application 5.10: *(Orthogonal groups in physics)*

Orthogonal groups $\mathrm{SO}(n)$ are used both as global and as gauge symmetries. The group $\mathrm{SO}(10)$ is a prominent unification group for the standard model of particle physics. The group $(\mathrm{S})\mathrm{O}(n-1,1)$ is the Lorentz group in n dimensions which underlies n-dimensional relativistic theories in much the same way $(\mathrm{S})\mathrm{O}(3,1)$ underlies four-dimensional covariant physics. The groups $\mathrm{SO}(p,q)$ also make an appearance as string duality symmetries.

Lie algebra

Following the usual approach and writing $A = \mathbb{1}_n + T + \cdots$ and inserting into $A^T \eta A = \eta$ leads to $T = -\eta T^T \eta$ so that the Lie algebra is given by

$$\mathrm{so}(p,q) = \{T \in \mathrm{End}(\mathbb{R}^n) \,|\, T = -\eta T^T \eta\} = \mathrm{Span}(\sigma_{\mu\nu}) \quad \Rightarrow \quad \dim(\mathrm{so}(p,q)) = \frac{1}{2} n(n-1) \,,$$

where the basis matrices $\sigma_{\mu\nu}$ are defined exactly as their Lorentz group counterparts (Eq. (5.9)) and also satisfy the same commutation relations, Eq. (5.10). The Cartan sub-algebra is spanned by the matrices $(\sigma_{12}, \sigma_{34}, \ldots, \sigma_{2m-1,2m})$ which contain non-trivial 2×2 blocks along the diagonal, with all other entries zero. For example, for $\mathrm{so}(n)$, these matrices are of the form

$$\sigma_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \,, \quad \sigma_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \,, \quad \ldots \tag{5.140}$$

Of course, the number of 2×2 blocks which fit along the diagonal equals m both for the even case, $n = 2m$, and the odd case, $n = 2m+1$, so that $\mathrm{rk}(\mathrm{so}(2m)) = m$ and $\mathrm{rk}(\mathrm{so}(2m+1)) = m$. This means that odd and even dimensions are quite different and have to be discussed separately. The effects of the signature of η are wiped out by complexification, just as in the unitary case, so

$$\mathrm{so}(p,q)_{\mathbb{C}} = \mathrm{so}(n)_{\mathbb{C}} =: \left\{ \begin{array}{lll} D_m & \text{for} & n = 2m \\ B_m & \text{for} & n = 2m+1 \end{array} \right. \tag{5.141}$$

with B_m and D_m the standard names for the complexified Lie algebras for the odd- and even-dimensional cases, respectively.

Cartan-Weyl formalism

This story is quite analogous to the one for A_n, so we will be concise.

Cartan decomposition of B_m and D_m

Analysing these algebras efficiently requires good conventions for generators - otherwise calculations can descend into a mess. It turns out, working with signature $(p,q) = (m,m)$ in the even case and $(p,q) = (m+1,m)$ in the odd case is the way to go, as well as using an off-diagonal metric

$$D_m \;:\quad \eta = \begin{pmatrix} 0 & \mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix} \,, \qquad B_m \;:\quad \eta = \begin{pmatrix} 0 & \mathbb{1}_m & 0 \\ \mathbb{1}_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{5.142}$$

This form of the metric results from a basis transformation

$$P^T\mathrm{diag}(\mathbb{1}_m, -\mathbb{1}_m)P = \begin{pmatrix} 0 & \mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix} \qquad \text{where} \qquad P = \frac{1}{2}\begin{pmatrix} \mathbb{1}_m & \mathbb{1}_m \\ \mathbb{1}_m & -\mathbb{1}_m \end{pmatrix} . \tag{5.143}$$

With this choice for the metric, the condition $T = -\eta T^T \eta$ leads to Lie-algebra elements of the form

$$D_m \;:\quad T = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} , \qquad B_m \;:\quad T = \begin{pmatrix} A & B & E \\ C & -A^T & F \\ -F^T & -E^T & 0 \end{pmatrix} \tag{5.144}$$

where A, B, C, D are $m \times m$ matrices with $B = -B^T$ and $C = -C^T$ and E, F are n-component vectors. One of the advantages of working in this basis is that the elements of the Cartan are actually diagonal matrices (unlike the $\sigma_{2k,2k+1}$ matrices considered earlier) and we can write down the Cartan subalgebra as

$$\mathcal{H} = \left\{ \sum_{i=1}^m b_i H_i \,|\, b_i \in \mathbb{C} \right\} \quad \text{where} \quad H_i = E_{ii} - E_{m+i,m+i} \;, \tag{5.145}$$

and E_{ij} are the standard unit matrices. As in the unitary case, we define the dual basis (L_i) with $L_i(H_j) = \delta_{ij}$, so that

$$\mathcal{H}' = \left\{ \sum_{i=1}^m l_i L_i \,|\, l_i \in \mathbb{C} \right\} \; . \tag{5.146}$$

A basis for the off-diagonal parts in the matrices (5.144) can also be constructed from the standard unit matrices and this leads to the following list of generators and roots.

algebra	generator	root	constraint	part in Eq. (5.144)
B_m and D_m	$X_{ij} = E_{ij} - E_{m+j,m+i}$	$L_i - L_j$	$i \neq j$	A
B_m and D_m	$Y_{ij} = E_{i,m+j} - E_{j,m+i}$	$L_i + L_j$	$i < j$	B
B_m and D_m	$Z_{ij} = E_{m+i,j} - E_{m+j,i}$	$-L_i - L_j$	$i < j$	C
B_m only	$U_i = E_{i,2m+1} - E_{2m+1,m+i}$	L_i		E
B_m only	$V_i = E_{m+i,2m+1} - E_{2m+1,i}$	$-L_i$		F

This means for D_m we have the Cartan-Weyl decomposition and root space

$$\begin{aligned} D_m &= \mathcal{H} \oplus \textstyle\bigoplus_{i\neq j} \mathbb{C}X_{ij} \oplus \bigoplus_{i<j} \mathbb{C}Y_{ij} \oplus \bigoplus_{i<j} \mathbb{C}Z_{ij} \\ \Delta &= \{\pm L_i \pm L_j \,|\, i \neq j\} \end{aligned} \tag{5.147}$$

and for B_m we have

$$\begin{aligned} B_m &= \mathcal{H} \oplus \textstyle\bigoplus_{i\neq j} \mathbb{C}X_{ij} \oplus \bigoplus_{i<j} \mathbb{C}Y_{ij} \oplus \bigoplus_{i<j} \mathbb{C}Z_{ij} \oplus \bigoplus_i \mathbb{C}U_i \oplus \bigoplus_i \mathbb{C}V_i \\ \Delta &= \{\pm L_i \pm L_j \,|\, i \neq j\} \cup \{\pm L_i\} \end{aligned} \; . \tag{5.148}$$

Killing form

The Killing form can be determined by a sum over the roots from Theorem 4.6 (v), just as we did for the A_n algebras, and this leads to

$$\Gamma\left(\sum_i b_i H_i, \sum_j \tilde{b}_j H_j\right) = N(n) \sum_i b_i \tilde{b}_i \quad \text{where} \quad N(n) = \begin{cases} 2n-2 & \text{for} \quad n = 2m+1 \\ 2n-4 & \text{for} \quad n = 2m \end{cases}$$

For an element $L = \sum_i l_i L_i \in \mathcal{H}'$ the dual is given by $H_L = \frac{1}{N(n)} \sum_i l_i H_i$ and, hence, the Killing form on the dual space is

$$\Gamma\left(\sum_i l_i L_i, \sum_j \tilde{l}_j L_j\right) = \frac{1}{N(n)} \sum_i l_i \tilde{l}_i \; . \tag{5.149}$$

To distinguish between positive and negative roots we introduce $\ell \sim (\ell_1, \ldots, \ell_m)$ with $\ell_1 > \ell_2 > \cdots > \ell_m > 0$, so that

$$\ell\left(\sum_i l_i L_i\right) = \sum_i \ell_i l_i \; . \tag{5.150}$$

Simple roots and Cartan matrix for D_m

From Eqs. (5.147) and (5.150) the positive roots are

$$\Delta_+ = \{L_i + L_j \,|\, i < j\} \cup \{L_i - L_j \,|\, i < j\} \; . \tag{5.151}$$

and for the simple positive roots we can select

$$\alpha_i = L_i - L_{i+1} \; , \quad i = 1, \ldots, m-1 \; , \qquad \alpha_m = L_{m-1} + L_m \; . \tag{5.152}$$

For $\lambda = \sum_i \lambda_i L_i$ it is then straightforward to work out the Killing forms

$$(\alpha_i, \alpha_i) = \frac{2}{N(n)} \; , \quad i = 1, \ldots, m \; , \qquad (\lambda, \alpha_i) = \frac{1}{N(n)} \left\{ \begin{array}{ll} \lambda_i - \lambda_{i+1} & \text{for} \quad i = 1, \ldots, m-1 \\ \lambda_{m-1} + \lambda_m & \text{for} \quad i = m \end{array} \right.$$

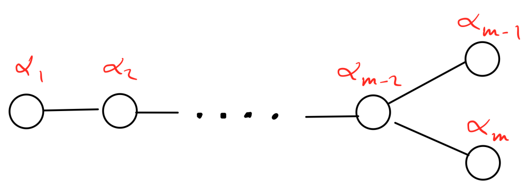
so that the Dynkin label of λ is

$$a_j = \frac{2(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} = \left\{ \begin{array}{ll} \lambda_j - \lambda_{j+1} & \text{for} \quad j = 1, \ldots, m-1 \\ \lambda_{m-1} + \lambda_m & \text{for} \quad j = m \end{array} \right. \quad \in \quad \mathbb{Z} \; . \tag{5.153}$$

Note that all simple positive roots have the same length. Inserting $\lambda = \alpha_i$ into this formula gives the entries of the Cartan matrix which reads

$$A(D_m) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix} \; . \tag{5.154}$$

This is very similar to the A_n Cartan matrix except for the entries in the right-lower 3×3 block. The Dynkin diagram which represents this matrix (following the earlier rules for how to associate Dynkin diagrams to Cartan matrices) is



Simple roots and Cartan matrix for B_m

From Eq. (5.148) and and (5.150) we have the positive roots

$$\Delta_+ = \{L_i + L_j \,|\, i < j\} \cup \{L_i - L_j \,|\, i < j\} \cup \{L_i\} \tag{5.155}$$

and due to the additional positive roots L_i, compared to the D_m case, the set of positive simple roots is now

$$\alpha_i = L_i - L_{i+1}\ , \quad i = 1, \ldots, m-1\ , \qquad \alpha_m = L_m\ . \tag{5.156}$$

For the length of these roots we find from Eq. (5.149) that

$$(\alpha_i, \alpha_i) = \frac{1}{N(n)} \left\{ \begin{array}{lll} 2 & \text{for} \quad i = 1, \ldots, m-1 & \text{(longer roots)} \\ 1 & \text{for} \quad i = m & \text{(shorter root)} \end{array} \right. \tag{5.157}$$

and for the Dynkin label of a weight $\lambda = \sum_i \lambda_i L_i$ we have

$$a_j = \frac{2(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} = \left\{ \begin{array}{ll} \lambda_i - \lambda_{i+1} & \text{for} \quad i = 1, \ldots, m-1 \\ 2\lambda_m & \text{for} \quad i = m \end{array} \right. \in \mathbb{Z}\ . \tag{5.158}$$

Inserting the simple roots $\lambda = \alpha_i$ into this result leads to the Cartan matrix

$$A(B_m) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -2 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \tag{5.159}$$

with associated Dynkin diagram

This is our first example of an algebra with different lengths for the roots and this is encoded in the Dynkin diagram by an empty (filled) node for the longer (shorter) roots.

Weight lattice

To find the weight lattice we should determine all $\lambda \in \mathcal{H}'$ for which the expressions (5.153) for D_m and (5.158) for B_m are integer for all α_i. For both D_m and B_m this leads to the same result, namely

$$\Lambda_W = \mathbb{Z}(L_1, \ldots, L_m, \alpha) \quad \text{where} \quad \alpha = \frac{1}{2}(L_1 + \cdots + L_m)\ . \tag{5.160}$$

Representations of D_m

Inverting the Eqs. (5.153) which relates a weight λ to its Dynkin label $(a_1, \ldots, a_m)$ we have

$$\lambda = (a_1 + \cdots a_{m-2})L_1 + (a_2 + \cdots + a_{m-2})L_2 + \cdots + a_{m-2}L_{m-2} + a_{m-1}\beta + a_m\alpha \qquad (5.161)$$

where

$$\alpha = \frac{1}{2}(L_1 + \cdots + L_m) \;, \qquad \beta = \frac{1}{2}(L_1 + \cdots + L_{m-1} - L_m) \;. \qquad (5.162)$$

The first $m-2$ terms in this expression are in complete analogy with the corresponding result (5.82) for A_n. Hence, for this part we can use the same tensor construction, starting with the fundamental representation acting on a vector space $V \cong \mathbb{C}^m$, in order to obtain the irreps. More precisely, the D_m irrep with highest weight Dynkin label $(a_1, \ldots, a_m)$ is contained in the tensor

$$S^{a_1}V \otimes S^{a_2}(\wedge^2 V) \otimes \cdots \otimes S^{a_{m-2}}(\wedge^{m-2}V) \otimes S^{a_{m-1}}\Gamma_\alpha \otimes S^{a_m}\Gamma_\beta \;, \qquad (5.163)$$

where Γ_α and Γ_β are, as yet, unknown and to be constructed representations with highest weights α and β and corresponding highest weight Dynkin labels $(0, \ldots, 0, 1)$ and $(0, \ldots, 0, 1, 0)$. In fact, Γ_α and Γ_β are the left- and right-handed Weyl spinor representations which we will construct soon.

Representations of B_m

We can proceed in analogy with the D_m case, except now we should invert the relations (5.158) which gives the weight

$$\lambda = (a_1 + \cdots a_{m-1})L_1 + (a_2 + \cdots + a_{m-1})L_2 + \cdots + a_{m-1}L_{m-1} + a_m\alpha \qquad (5.164)$$

where

$$\alpha = \frac{1}{2}(L_1 + \cdots + L_m) \;. \qquad (5.165)$$

The B_m irrep with highest weight Dynkin label $(a_1, \ldots, a_m)$ is then contained in the tensor

$$S^{a_1}V \otimes S^{a_2}(\wedge^2 V) \otimes \cdots \otimes S^{a_{m-1}}(\wedge^{m-1}V) \otimes S^{a_m}\Gamma_\alpha \;, \qquad (5.166)$$

where Γ_α is a representation with highest weight α and corresponding highest weight Dynkin label $(0, \ldots, 0, 1)$. This is the Dirac spinor representation (no Weyl spinors in odd dimensions $n = 2m + 1$ as we will see) which will be constructed below.

While we have now constructed all D_m and B_m representations the tensors (5.163) and (5.166) are not, in general, irreducible, just as in the A_n case, but they do contain the irrep with highest weight Dynkin label $(a_1, \ldots, a_m)$. For the A_n case the refinement of symmetrisation/anti-symmetrisation facilitated by Young tableaux led to irreducible tensors. These methods can also be employed here and they typically lead to a reduction in the size of the tensor but, unlike in the A_n case, not necessarily to an irrep. We will not pursue this further since the general methods to be developed later will cover these cases. However, we should complete the discussion by constructing the so far unknown spinor representations Γ_α and Γ_β.

Spinor representations

Due to the importance of spinors for physics in d space-time dimensions, we will carry this discussion out for a metric $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$ with Lorentzian signature $(p, q) = (d-1, 1)$, so consider the group $\mathrm{SO}(d-1, 1)$. We will also use a somewhat low-key approach, common in physics, rather than roll out the full formalism of Clifford algebras. It is useful to introduce the numbers

$$k = \left[\frac{d}{2}\right] - 1 = \begin{cases} \frac{d-2}{2} & \text{for} \quad d \text{ even} \\ \frac{d-3}{2} & \text{for} \quad d \text{ odd} \end{cases} . \tag{5.167}$$

Gamma matrices

A set of d matrices γ_μ, where $\mu = 0, \ldots, d-1$, of size $2^{k+1} \times 2^{k+1}$ is called a set of **gamma matrices** iff

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{5.168}$$

(Here the bracket $\{\cdot, \cdot\}$ denotes the anti-commutator, defined as $\{A, B\} := AB + BA$.) The relations (5.168) imply that different gamma matrices anti-commute, $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$, for $\mu \neq \nu$, and that they square to the identity (or minus the identity in the time direction), $\gamma_\mu^2 = \eta_{\mu\mu}\mathbb{1}$. The algebra generated by these matrices is also referred to as a **Clifford algebra**.

Construction of gamma matrices

Of course we should demonstrate that gamma matrices exist in all dimensions d and we do this by induction in d, starting with $d = 2$. To keep track of dimensions we denote the gamma matrices in d dimensions by $\gamma_\mu^{(d)}$. The gamma matrices $\gamma_\mu^{(2)}$ in $d = 2$ dimensions are 2×2 matrices (since $k = 0$ form Eq. (5.167)) and they are easy to construct starting from the Pauli matrices. A viable choice is

$$\gamma_0^{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \,, \qquad \gamma_1^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \,. \tag{5.169}$$

Note that these two matrices do indeed anti-commute, while $(\gamma_0^{(2)})^2 = -\mathbb{1}_2$ and $(\gamma_i^{(2)})^2 = \mathbb{1}_2$, as required.

For the next step we assume that gamma matrices $\gamma_\mu^{(d)}$ in even d dimensions have been constructed and we construct the gamma matrices in dimension $d+2$. A possible choice is

$$\gamma_\mu^{(d+2)} = \gamma_\mu^{(d)} \times \sigma_3,\ \mu = 0, \ldots, d\,, \qquad \gamma_d^{(d+2)} = \mathbb{1} \times \sigma_1\,, \qquad \gamma_{d+1}^{(d+2)} = \mathbb{1} \times \sigma_2\,. \tag{5.170}$$

Note that tensoring with the 2×2 Pauli matrices doubles the size of the matrices, precisely as required from Eq. (5.167). That these matrices satisfy the correct relations (5.168) follows from the fact that $\gamma_\mu^{(d)}$ do, that the Pauli matrices anti-commute and square to $\mathbb{1}_2$ and the second property (1.32) of the Kronecker product. This provides us with gamma matrices in all even dimensions.

To obtain gamma matrices in odd dimensions, we start with an even dimension d and gamma matrices $\gamma_\mu^{(d)}$ and construct gamma matrices in $d+1$ dimensions. To this end we define the matrix

$$\gamma = i^{-k}\gamma_0^{(d)}\gamma_1^{(d)} \cdots \gamma_{d-1}^{(d)} \quad \Rightarrow \quad \{\gamma, \gamma_\mu^{(d)}\} = 0\,, \quad \gamma^2 = \mathbb{1}\,, \tag{5.171}$$

the analogue of γ_5 in four dimensions. Clearly, this matrix anti-commutes with all gamma matrices since $\gamma_\mu^{(d)}$ commutes with itself which leaves an odd number, $d-1$, of anti-commutations and, hence, an overall minus sign. Also it is clear that γ must square to $\pm\mathbb{1}$ (just anti-commute and use that the $\gamma_\mu^{(d)}$ square to $\pm\mathbb{1}$). The factor in Eq. (5.171) is precisely such that γ squares to $+\mathbb{1}$. Note that the gamma matrices in an even dimension d and in $d+1$ have the same size (see Eq. (5.167)) so for the gamma matrices in $d+1$ dimensions we can take

$$\gamma_\mu^{(d+1)} = \gamma_\mu^{(d)} , \quad \mu = 0, \ldots, d-1 , \qquad \gamma_d^{(d+1)} = \gamma . \tag{5.172}$$

Generators of $\mathrm{so}(d-1,1)$

How are the gamma matrices related to representations of $\mathrm{so}(d-1,1)$? If we define

$$\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{5.173}$$

then a straightforward calculation, using only the anti-commutation relations (5.168), shows that the $\sigma_{\mu\nu}$ satisfy the commutation relations (5.10) of $\mathrm{so}(d-1,1)$ and, hence, define a representation of this algebra. This representation is referred to as the **Dirac spinor** representation in d dimensions. It acts on Dirac spinors $\psi \in \mathbb{C}^{2^{k+1}}$ as

$$\delta\psi = i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi \tag{5.174}$$

with parameters $\epsilon^{\mu\nu}$. The Dirac spinor in d dimensions is a representation with complex dimensions 2^{k+1}. As we will see it is reducible for d even (leading to the left- and right-handed Weyl spinors), but for d odd it is, in fact, irreducible (as a complex representation) and corresponds to the irrep Γ_α in Eq. (5.166).

Proposition 5.1. *For $d = 2m+1$, the matrices $\sigma_{\mu\nu}$ in Eq, (5.173) generate the spinor representation Γ_α.*

Proof. The idea of the proof is to identify the highest weight vector in the representation defined by the $\sigma_{\mu\nu}$ and to show that its weight equals α. Recall that elements of the Cartan are of the form

$$H = \begin{pmatrix} B & 0 & 0 \\ 0 & -B & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad B = \mathrm{diag}(b_1, \ldots, b_m) .$$

The other generators we need from the Cartan-Weyl decomposition (5.148) are the generators

$$U_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_i \\ e_i^T & 0 & 0 \end{pmatrix} ,$$

which correspond to the positive roots L_i. Of course these generators are written in a basis where the metric is off-diagonal so we should rotate back using the basis transformation P in Eq. (5.143) to obtain

$$\hat{H} = PHP = \begin{pmatrix} 0 & B & 0 \\ -B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sum_{i=1}^m b_i \sigma_{i,m+i}$$

$$\hat{U}_i = PU_iP = \begin{pmatrix} 0 & 0 & e_i \\ 0 & 0 & -e_i \\ -e_i^T & e_i^T & 0 \end{pmatrix} = \sigma_{i,2m+1} - \sigma_{m+i,2m+1}$$

The highest weight state v must satisfy $U_i v = 0$, for $i = 1, \ldots, m$ which translates to $\gamma_i \gamma_{m+i} v = v$. Hence

$$Hv = \sum_{i=1}^{m} b_i \sigma_{i,m+i} v = \frac{1}{2} \sum_{i=1}^{m} b_i \gamma_i \gamma_{m+1} v = \frac{1}{2} \sum_{i=1}^{m} b_i v \overset{(5.165)}{=} \alpha(H) v \ ,$$

so the weight of v is indeed α, as required. □

It is useful to think about the hermitian conjugate of the gamma matrices. From our recursive construction of γ_μ it is clear that $\gamma_0^\dagger = -\gamma_0$ and $\gamma_i^\dagger = \gamma_i$ for $i = 1, \ldots, d-1$. These equations can be conveniently summarised by writing

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \quad \Rightarrow \quad \sigma_{\mu\nu}^\dagger = -\gamma_0 \sigma_{\mu\nu} \gamma_0 \ . \tag{5.175}$$

Weyl spinors

In odd dimensions d the Dirac spinor is irreducible (as a complex representation). However, for even d the matrix γ in Eq. (5.171) is not among the gamma matrices, yet anti-commutes with them. This means that $[\sigma_{\mu\nu}, \gamma] = 0$, so γ commutes with the entire representation. Also, since $\gamma^2 = \mathbb{1}$ and $\mathrm{tr}(\gamma) = 0$ (this follows from anti-commutation and cyclicity of the trace) we can define projectors

$$P_{L,R} = \frac{1}{2}(\mathbb{1} \pm \gamma) \qquad \Rightarrow \qquad P_{L,R}^2 = P_{L,R} \ , \quad \mathrm{tr}(P_{L,R}) = 2^k \tag{5.176}$$

so we have two projectors onto sub-representations, each with the (complex) dimension 2^k (so half the complex dimension $2^{k+1}$of the Dirac spinor). These representations are called left- and right-handed **Weyl spinor** representations and the spinors $\psi_{L,R} = P_{L,R}\psi$ are called left- and right-handed Weyl spinors. The left- or right-handedness of the spinor is also referred to as the **chirality** of the spinor. These are, in fact, irreducible representations which correspond to the representations Γ_α and Γ_β in Eq. (5.163). The proof is similar to the one in Prop. 5.1.

Complex conjugation and Majorana spinors

We would like to understand if we can impose a reality condition on spinors, so if we can reduce the complex spinor representations obtained so far to real representations. The first useful observation is that the gamma matrices and their complex conjugates (which also satisfy Eq. (5.168)) are related by a basis transformation with

$$B_1 = \gamma_3 \gamma_5 \cdots \ , \qquad B_2 = \gamma B_1 \tag{5.177}$$

such that (for d even and $\mu = 0, \ldots, d-1$)

$$\begin{array}{rclcrcl} B_1 \gamma_\mu B_1^{-1} & = & (-1)^k \gamma_\mu^* & \qquad & B_2 \gamma_\mu B_2^{-1} & = & (-1)^{k+1} \gamma_\mu^* \\ B_1 \gamma B_1^{-1} & = & (-1)^k \gamma^* & & B_2 \gamma B_2^{-1} & = & (-1)^k \gamma^* \end{array} \tag{5.178}$$

Translating this to the generators $\sigma_{\mu\nu}$ it follows that

$$B \sigma_{\mu\nu} B^{-1} = -\sigma_{\mu\nu}^* \quad \text{where} \quad B = \left\{ \begin{array}{ll} B_1 \text{ or } B_2 & \text{for} \quad d \text{ even} \\ B_1 & \text{for} \quad d \text{ odd} \end{array} \right. \ . \tag{5.179}$$

Note that for even dimensions we can use either B_1 or B_2 (since we only need the γ_μ for $\mu = 0, \ldots, d-1$) while for d odd only B_1 works, since we need both γ_μ and γ but the signs in Eq. (5.178) differ for B_2.

For a Dirac spinor ψ in d dimensions we can now define the **charge conjugate** spinor

$$\psi^c = C\psi := B^{-1}\psi^* \ , \tag{5.180}$$

with the matrix B from Eq. (5.179). This is basically just the complex conjugate spinor but the matrix B is included to account for the fact that the generators $\sigma_{\mu\nu}$ might not be real. The point is that charge conjugation commutes with the so$(d-1,1)$ transformation as the following short calculation shows.

$$[i\sigma_{\mu\nu}, C]\psi = i\sigma_{\mu\nu}B^{-1}\psi^* - B^{-1}(i\sigma_{\mu\nu}\psi)^* = i \underbrace{\sigma_{\mu\nu}B^{-1}}_{=-B^{-1}\sigma^*_{\mu\nu}} \psi^* + iB^{-1}\sigma^*_{\mu\nu}\psi^* = 0 \ .$$

What happens if charge conjugation is applied twice? Since $(\psi^c)^c = (B^{-1}\psi^*)^c = (B^*B)^{-1}\psi$ and, on the other hand, explicit calculations, using the matrices (5.177), leads to

$$\begin{aligned} B_1^*B_1 &= (-1)^{k(k+1)/2}\mathbb{1} \\ &= \mathbb{1} \text{ for } k = 0,3 \bmod 4 \end{aligned} \qquad \begin{aligned} B_2^*B_2 &= (-1)^{(k(k-1)/2}\mathbb{1} \\ &= \mathbb{1} \text{ for } k = 0,1 \bmod 4 \end{aligned}$$

we see that applying charge conjugation twice is either the identity or minus the identity. The former is the case if k satisfies the above constraints and translating these into constraints on the dimension d, via Eq. (5.167), we learn that a matrix B with $B^*B = \mathbb{1}$ is available iff $d = 0,1,2,3,4 \bmod 8$. In such dimensions we can define a **Majorana spinor** by imposing on a Dirac spinor ψ the constraint

$$\psi^c = \psi \ . \tag{5.181}$$

(In the other case, when $(\psi^c)^c = -\psi$, Eq. (5.181) leads to $\psi = -\psi^c$ which implies $\psi = 0$.) This defines a 2^{k+1} real-dimensional representation of so$(d-1,1)$.

Majorana-Weyl spinors

Can Majorana and Weyl conditions be imposed simultaneously on a Dirac spinor? Clearly, this can only be attempted if the dimension d is even (so that Weyl spinors exist) and $d = 0,1,2,3,4 \bmod 8$ so that Majorana spinors exist and, hence, only in dimensions $d = 0,2,4 \bmod 8$. In addition, in order to be able to impose both conditions, we require that $[\gamma, C] = 0$. This implies

$$[\gamma, C]\psi = \gamma B^{-1}\psi^* - B^{-1}\gamma^*\psi^* \overset{!}{=} 0 \quad \Leftrightarrow \quad B\gamma B^{-1} = \gamma^* \quad \Leftrightarrow \quad k \text{ even}$$

The dimensions $d = 0,2,4 \bmod 8$ for which k is even are precisely $d = 2 \bmod 8$ so only in those dimensions can Majorana-Weyl spinors be defined. They are spinors with 2^k real components.

Spinors in diverse dimensions

We have seen that there is considerable structure in spinors, with the existence of certain types of spinors related to the dimension d. The results, for the case of Lorentzian signature [9], are summarised in Table 5.1.

[9] Weyl spinors always exist in even dimensions for any signature of the metric η. However, the existence of Majorana (and Majorana-Weyl) spinors depends on the signature and our results here are for the Lorentzian signature.

spinor type	real dimension	exist in dimensions
Dirac	2^{k+2}	d
Weyl	2^{k+1}	d even
Majorana	2^{k+1}	$d = 0, 1, 2, 3, 4 \mod 8$
Majorana-Weyl	2^k	$d = 2 \mod 8$

Table 5.1: Types of spinors, their dimensions and the Lorentzian space-time dimensions d they exist in. Here, k is defined in terms of d by Eq. (5.167).

Application 5.11: *(Lorentz-invariant spinor terms in d dimensions)*
The above result on spinors can be used to construct Lorentz invariant theories for spinors in d (Lorentzian) dimensions. To this end it is useful to define, for a Dirac spinor ψ, the conjugate spinor $\bar{\psi} = \psi^\dagger \gamma_0$. (This is just the hermitian conjugate where the additional γ_0 factor is included to take care of the negative signature in the time direction, as suggested by Eq. (5.175).) As an example, we would like to show that the term $\bar{\psi}\psi$ is Lorentz-invariant, using the transformation $\delta\psi = i\epsilon^{\mu\nu}\sigma_{\mu\nu}$.

$$\begin{aligned} \delta(\bar{\psi}\psi) &= \overline{\delta\psi}\,\psi + \bar{\psi}\,\delta\psi = (i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi)^\dagger\gamma_0\psi + \psi^\dagger\gamma_0(i\epsilon^{\mu\nu}\sigma_{\mu\nu})\psi \\ &= i\epsilon^{\mu\nu}\left(-\psi^\dagger\sigma^\dagger_{\mu\nu}\gamma_0\psi + \psi^\dagger\gamma_0\sigma_{\mu\nu}\psi\right) \overset{(5.175)}{=} 0 \end{aligned}$$

Application 5.12: *(Supersymmetry in diverse dimensions)*
Supersymmetry is a symmetry parametrised by spinors and is quite unlike the symmetries (= groups) discussed here. We will not discuss supersymmetry in any detail but merely capitalise on the insights we can gain from the above classification of spinors in diverse dimensions. The minimal amount of supersymmetry in a given dimension d, also referred to as $N = 1$ supersymmetry, is parametrised by the smallest spinor available in that dimension. Let us consider this in a few relevant dimensions (using the results from Table (5.1)) by writing down the spinor types and their real dimensions.

d	k	Dirac	Weyl	Majorana	Majorana-Weyl
4	1	8	4	4	-
5	1	8	-	-	-
⋮	⋮	⋮	⋮	⋮	⋮
10	4	64	32	32	16
11	4	64	-	32	-

The smallest spinor in $d = 4$ dimensions has 4 real components (Weyl or Majorana) and it parametrises $N = 1$ supersymmetry in $d = 4$ which is the framework considered for supersymmetric models of particle physics. Note that the minimal spinor in $d = 5$ dimensions has 8 real components, so minimal $N = 1$ supersymmetry in $d = 5$ has twice as many supercharges as $N = 1$ supersymmetry in $d = 4$. The minimal spinor in $d = 10$ dimensions is the Majorana-Weyl spinor with 16 real components. It parametrises the minimal $N = 1$ supersymmetry in 10-dimensions. The effective ten-dimensional supergravity theories for the heterotic and the type I string theories are precisely of this type. Non-minimal $N = 2$ supersymmetry in $d = 10$ is parametrised by two Majorana-Weyl spinors and, hence, has 32 supercharges. There are two choices here, namely to use two Majorana-Weyl spinors with the same or with different chiralities. The former option is realised for the type IIA supergravity (the effective ten-dimensional supergravity of the type IIA superstring) and the latter for the type IIB supergravity (the effective ten-dimensional supergravity of the type IIB superstring). Finally, the minimal $N = 1$ supersymmetry in $d = 11$ has 32 supercharges and this is realised by 11-dimensional supergravity, the low-energy effective theory of M-theory.

Chapter 6

Classification and Dynkin formalism

In this chapter, we would like to pull together the various strands of our discussion by classifying simple, complex Lie algebras and by describing their representations in a systematic way. The approach relies on understanding roots and weights, following on from where we left off in Chapter 4, and it puts some of the examples developed in the previous chapter into a systematic context.

6.1 Classification of simple, complex Lie algebras

The classification of simple, complex Lie algebras is based on studying the allowed root systems for such algebras with the goal of finding all allowed root diagrams. We are working in a complex, semi-simple Lie algebra $\mathcal{L}$ with root space Δ and Cartan-Weyl decomposition

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_{\alpha\in\Delta} \mathcal{L}_\alpha \; . \tag{6.1}$$

Geometry of roots

To classify root systems we need to study their geometry and in this context the following quantities (for $\alpha, \beta \in \Delta$) are helpful.

$$||\alpha|| = (\alpha,\alpha)^{1/2} \; , \qquad \cos\theta_{\alpha\beta} = \frac{(\alpha,\beta)}{||\alpha||\,||\beta||} \; , \qquad n_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \tag{6.2}$$

Evidently, these are the length $||\alpha||$ of a root α and the angle $\theta_{\alpha\beta}$ between two roots. We know that $n_{\beta\alpha}$ must be integer (see Eq. (4.38)). Between those quantities, we have the obvious relations

$$n_{\beta\alpha} = n_{\alpha\beta}\frac{||\beta||^2}{||\alpha||^2} \; , \quad n_{\beta\alpha} = 2\frac{||\beta||}{||\alpha||}\cos\theta_{\alpha\beta} \qquad \Rightarrow \qquad n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\theta_{\alpha\beta} \in [0,4] \; . \tag{6.3}$$

The equation on the right puts a strong constraint on $n_{\beta\alpha}$ which, in addition to being an integer, must also satisfy $n_{\alpha\beta}n_{\beta\alpha} \in [0,4]$. For definiteness, let us assume that $||\beta|| \geq ||\alpha||$, so that $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$. If $n_{\alpha\beta}n_{\beta\alpha} = 4$ then $\cos\theta_{\alpha\beta} = \pm 1$ which means that $\beta = \pm\alpha$. Apart from this trivial case, the solutions for $n_{\alpha\beta}$ are listed in Table 6.1 (with the other quantities calculated from Eqs. (6.3)). This shows that two roots can be parallel or anti-parallel (the trivial case $n_{\alpha\beta}n_{\beta\alpha} = 4$) or else must form one of seven possible angles, as in Table 6.1, in each case with a specific ratio $||\beta||/||\alpha||$ of the two lengths. Hence, the basic building blocks for

$n_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$n_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\|\|\beta\|\|/\|\|\alpha\|\|$	$\sqrt{3}$	$\sqrt{2}$	1	-	1	$\sqrt{2}$	$\sqrt{3}$
$\cos\theta_{\alpha\beta}$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
$\theta_{\alpha\beta}$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$

Table 6.1: Possible angles and length ratios between roots.

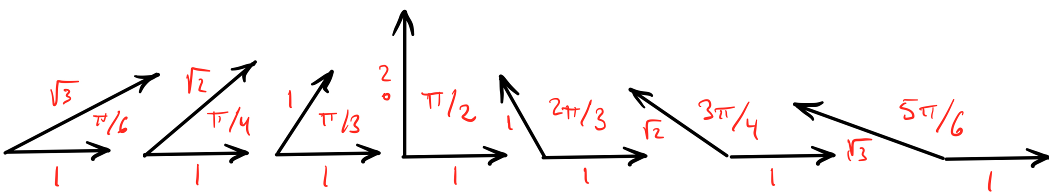


Figure 6.1: The building blocks for root diagrams.

root diagrams are the ones shown in Fig. 6.1. The following is a useful piece of terminology to describe root systems.

Definition 6.1. *For two roots $\alpha, \beta \in \Delta$ the α-**string through** β is the list of all elements in $\Delta \cup \{0\}$ of the form*

$$\beta - p\alpha,\ \beta - (p-1)\alpha,\ \ldots\ , \beta - \alpha,\ \beta,\ \beta + \alpha,\ \ldots\ , \beta + (q-1)\alpha,\ \beta + q\alpha\ . \tag{6.4}$$

Theorem 6.1. *For $\alpha \neq \pm\beta$ the α-strings through β has the following properties.*
(i) The α-string through α consists of $-\alpha$, 0, α.
(ii) $p - q = n_{\beta\alpha}$ with p, q as in Eq. (6.4).
(iii) $p + q \leq 3$ so the string has at most length four.
(iv) If $(\beta, \alpha) > 0$ then $\alpha - \beta$ is a root and if $(\beta, \alpha) < 0$ then $\alpha + \beta$ is a root.

Proof. (i) This was already shown in Theorem 4.6 (v).
(ii) Recall that $(H_\alpha, E_\alpha, E_{-\alpha})$ span an $\mathrm{su}(2)_\mathbb{C}$ subalgebra with $[H_\alpha, E_{\pm\alpha}] = \pm(\alpha,\alpha)E_{\pm\alpha}$. The α-string through β must form an irrep under this $\mathrm{su}(2)_\mathbb{C}$ with the roots at the end of the string transforming as

$$\mathrm{ad}(H_\alpha)(E_{\beta-p\alpha}) = (\beta - p\alpha, \alpha)E_{\beta-p\alpha}\ , \qquad \mathrm{ad}(H_\alpha)(E_{\beta+q\alpha}) = (\beta + q\alpha, \alpha)E_{\beta+q\alpha}$$

The eigenvalues $(\beta - p\alpha, \alpha)$ and $(\beta + q\alpha, \alpha)$ must correspond to a spin $\pm j \in \mathbb{Z}/2$ or, more precisely, we must have

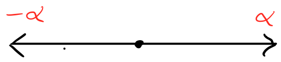
$$(\beta - p\alpha, \alpha) = -(\alpha,\alpha)j\ , \quad (\beta + q\alpha, \alpha) = (\alpha,\alpha)j \qquad \Rightarrow \qquad p - q = n_{\beta\alpha}\ .$$

(iii) Set $\beta' = \beta - p\alpha$ (the left-hand end of the string) and focus on the α string through β', given by β', $\beta' + \alpha$, $\cdots$, $\beta' + q\alpha$. Applying (ii) to this string immediately gives $q = |n_{\beta'\alpha}| \leq 3$.
(iv) From (ii) we have $p - q = n_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. So if $(\beta, \alpha) > 0$ then $p > 0$ and $\alpha - \beta$ is a root. On the other hand, if $(\beta, \alpha) < 0$ then $q > 0$ and $\alpha + \beta$ is a root. $\square$

Classification by drawing

With the information about the structure of root diagrams we have collected we can attempt a classification by drawing all diagrams. This is only really practical in dimensions one and two, so for algebras with $\mathrm{rk}(\mathcal{L}) = 1$ and $\mathrm{rk}(\mathcal{L}) = 2$.

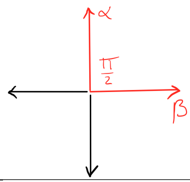
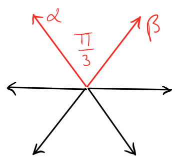
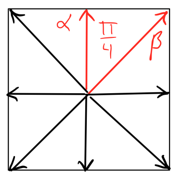
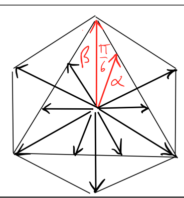
For $\mathrm{rk}(\mathcal{L}) = 1$ Theorem 6.1 (i) only leaves one possibility,



$$A_1 \cong \mathrm{su}(2)_{\mathbb{C}}$$

which is of course the root diagram for A_1.

Things become more interesting for $\mathrm{rk}(\mathcal{L}) = 2$. We can start by drawing two roots, α and β with (say) $||\beta|| \geq ||\alpha||$, choosing one of the angles $\leq \pi/2$ available from Table 6.1. The negatives are always also roots, so $-\alpha$ and $-\beta$ can immediately be added to the diagram. Thereafter, the basic rules for drawing are provided by Theorem 6.1, in particular part (iv) which says that we should add the sum $\gamma + k\delta$, for $k = 1, \ldots, -n_{\gamma\delta}$ of two roots γ, δ to the diagram provided they form an angle $\theta_{\gamma\delta} > \pi/2$ (so that $(\gamma, \delta) < 0$).

angle	root diagram	algebra
$\theta = \pi/2$		$D_2 \cong A_1 \oplus A_1 \cong \mathrm{su}(2)_{\mathbb{C}} \oplus \mathrm{su}(2)_{\mathbb{C}}$
$\theta = \pi/3$		$A_2 \cong \mathrm{su}(3)_{\mathbb{C}}$
$\theta = \pi/4$		$C_2 \cong B_2 \cong \mathrm{so}(5)_{\mathbb{C}}$
$\theta = \pi/6$		G_2

We have already encountered the first three of these rank two algebras. The last one, G_2, is new and is one of the exceptional Lie algebras which will emerge from the systematic classification below.

Simple positive roots

Classification of simple Lie algebras by drawing their root diagrams becomes impractical for $\mathrm{rk}(\mathcal{L}) > 2$ so we need more sophisticated methods to proceed. The way forward is to focus on the simple positive roots, rather than the entire root system.

Lemma 6.1. *If $\alpha, \beta \in \Delta_+$ are positive simple roots we have the following statements.*
(i) $\alpha - \beta$ and $\beta - \alpha$ are not roots.
(ii) $(\alpha, \beta) \leq 0$.

Proof. (i) We can write $\alpha = \beta + (\alpha - \beta)$ and since α is simple we conclude that $\alpha - \beta \notin \Delta_+$ (otherwise we have written α as a sum of two positive roots which contradicts simplicity of α). Similarly, $\beta = \alpha + (\beta - \alpha)$ implies that $\beta - \alpha \notin \Delta_+$ and, hence $\alpha - \beta \notin \Delta_-$. Since $\Delta = \Delta_+ \cup \Delta_-$ we conclude that $\alpha - \beta \notin \Delta$, so it is not a root. Then its negative, $\beta - \alpha$, is also not a root.
(ii) From Theorem 6.1 (iv) we know $(\beta, \alpha) > 0$ implies that $\alpha - \beta$ is a root. Since part (i) of the present Lemma says that $\alpha - \beta$ is not a root we conclude that $(\beta, \alpha) \leq 0$. $\square$

Recall that for rank n we have n simple, positive roots $(\alpha_1, \ldots, \alpha_n)$ which form a basis of $\mathcal{H}'$. The previous Lemma says that simple positive roots α, β satisfy $n_{\beta\alpha} \leq 0$ and, therefore, for such roots only the possibilities in the four rightmost columns of Table 6.1 are relevant.

Dynkin diagrams

These four possible relationships between positive simple roots can be represented graphically, by assigning nodes to β and α and connecting them by $l_{\beta\alpha} := -n_{\beta\alpha}$ lines (assuming $||\beta|| \geq ||\alpha||$), as summarised in Table 6.2. The four diagrams in this table are the basic building

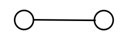
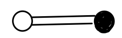
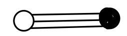
$\theta_{\beta\alpha}$	$n_{\beta\alpha} = -l_{\beta\alpha}$	$n_{\alpha\beta}$	β α
$\pi/2$	0	0	
$2\pi/3$	-1	-1	
$3\pi/4$	-2	-1	
$5\pi/6$	-3	-1	

Table 6.2: Basic building blocks of Dynkin diagrams for two positive simple roots α, β with $||\beta|| \geq ||\alpha||$, covering the four rightmost columns in Table 6.1. If $||\beta|| > ||\alpha||$, the open node represents the longer root β, and the filled node the shorter root α.

blocks of Dynkin diagrams. A Dynkin diagram for n simple roots $(\alpha_1, \ldots, \alpha_n)$ consists of n nodes, one for each root, connected by one of the four elements in the table. As we will see, only two root lengths are required in any Dynkin diagram, so using open and filled nodes will be sufficient to describe all cases.

The Cartan matrix can be inferred from the Dynkin diagram, by reading off the number of lines $l_{\alpha_i\alpha_j}$ between each pair of nodes (α_i, α_j) for $i \neq j$ and then combining the four cases in Table 6.2 with the definition of $n_{\beta\alpha}$ in Eq. (6.2). The result is

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{cases} 2 & \text{for } i = j \\ -l_{\alpha_i\alpha_j} & \text{for } i \neq j,\ ||\alpha_i|| \geq ||\alpha_j|| \\ -1 & \text{for } i \neq j,\ ||\alpha_i|| < ||\alpha_j||,\ l_{\alpha_i\alpha_j} \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.5}$$

Conversely, given a Cartan matrix A the Dynkin diagram can be immediately read off. From the first Eq. (6.3) we have

$$A_{ij} = \frac{||\alpha_i||^2}{||\alpha_j||^2} A_{ji}$$

and this can be used to determine the relative lengths of the roots and assign open and filled nodes accordingly. Then we should draw $l_{\alpha_i\alpha_j} = \max(-A_{ij}, -A_{ji})$ lines between nodes α_j and α_j for $i \neq j$.

Admissible Dynkin diagrams

We would like to classify all simple complex Lie algebras by drawing all Dynkin diagrams, so we have to think about which of these diagrams correspond to actual Lie algebras. In this context it is useful to observe that the angle between two roots β and α (as computed from the inner product on $\mathcal{H}'$) and the lines they are connected with in the Dynkin diagram are related by

$$\cos\theta_{\beta\alpha} = -\frac{\sqrt{l_{\beta\alpha}}}{2} \; . \tag{6.6}$$

This can be verified by just going through the four cases in Table 6.2. This relation constraints Dynkin diagrams and this motivates the following definition.

Definition 6.2. *A Dynkin diagram with n nodes is called* ***admissible*** *if there exist n linearly independent vectors $\alpha_1, \ldots, \alpha_n \in \mathcal{H}'$, associated to the n nodes, such that $\cos\theta_{\alpha_i\alpha_j} = -\sqrt{l_{\alpha_i\alpha_j}}/2$ for all $i \neq j$, where $\cos\theta_{\alpha_i\alpha_j}$ is computed from the inner product on $\mathcal{H}'$ and the number of lines, $l_{\alpha_i\alpha_j}$, is read off from the Dynkin diagram.*

In view of Eq. (6.6) all Dynkin diagrams associated to actual Lie algebras must be admissible, so this is a necessary condition for a Dynkin diagram to describe a Lie algebra. It is by no means clear that it is also sufficient but this is not really required for the purpose of classification. All we need is a sufficiently strong necessary condition which reduces the number of cases to a manageable amount. Perhaps surprisingly, the simple condition of admissibility is sufficiently strong in this sense, as the classification below will show.

Constraints on admissible diagrams

We will now proof a series of lemmas which constrain admissible Dynkin diagrams more and more to a point where we can just compile a list of the remaining admissible diagrams. We begin with a simple statement about sub-diagrams.

Lemma 6.2. *All sub-diagrams of admissible diagrams are admissible.*

Proof. This follows immediately from the definition of "admissible". □

The following is a strong upper bounds on the number of links in an admissible diagram.

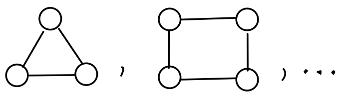
Lemma 6.3. *An admissible diagram with n nodes has at most $n - 1$ links (where two nodes connected by a single, double, or triple line count as one link).*

Proof. Define the vector $\alpha = \sum_{i=1}^{n} \frac{\alpha_i}{||\alpha_i||}$ which must be non-zero since the α_i are linearly independent. It follows that

$$0 < ||\alpha||^2 = \sum_{i,j=1}^{n} \frac{(\alpha_i, \alpha_j)}{||\alpha_i||\,||\alpha_j||} = n + 2 \sum_{i,j \text{ linked}} \frac{(\alpha_i, \alpha_j)}{||\alpha_i||\,||\alpha_j||} = n + 2 \sum_{i,j \text{ linked}} \cos\theta_{\alpha_i\alpha_j}$$
$$= n - \sum_{i,j \text{ linked}} \sqrt{l_{\alpha_i\alpha_j}} \leq n - \#\text{ links}$$

It follows that # links $< n$ which is the desired statement. $\square$

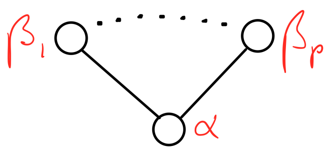
Remark 6.1. *Lemma 6.3 excludes all diagrams which are loops, such as*



From Lemma 6.2 this also excludes all diagrams which contain loops as sub-diagrams.

Lemma 6.4. *A node in an admissible diagrams is connected to at most three lines.*

Proof. Consider a root α of the admissible diagram and roots $\beta_1, \ldots, \beta_p$ of the diagram linked to α (by a single, double or triple line) so a configuration such as



Note that we must have $(\beta_i, \beta_j) = 0$ for all $i \neq j$. Otherwise, if $(\beta_i, \beta_j) \neq 0$, we have $l_{\beta_i\beta_j} \neq 0$ from Eq. (6.6) so that $(\alpha, \beta_i, \beta_j)$ would form a loop, a possibility excluded by the previous two lemmas. Let α_0 be the projection of α onto $\text{Span}(\beta_1, \ldots, \beta_p)$, so that $\gamma := \alpha - \alpha_0$ is orthogonal to all β_i, that is, $(\gamma, \beta_i) = 0$ for all $i = 1, \ldots, p$. Hence, the vectors $(\gamma, \beta_1, \ldots, \beta_p)$ are pairwise orthogonal and we can write

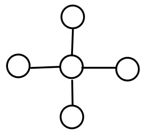
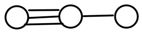
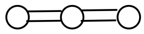
$$\alpha = \frac{(\gamma, \alpha)}{||\gamma||^2}\gamma + \sum_{i=1}^{p} \frac{(\beta_i, \alpha)}{||\beta_i||^2}\beta_i \qquad \Rightarrow \qquad ||\alpha||^2 = \frac{(\gamma, \alpha)^2}{||\gamma||^2} + \sum_{i=1}^{p} \frac{(\beta_i, \alpha)^2}{||\beta_i||^2}$$

Using the second equation (divided by $||\alpha||^2$) it follows that

$$\sum_{i=1}^{p} l_{\alpha\beta_i} = 4 \sum_{i=1}^{4} \cos^2\theta_{\alpha\beta_i} = 4 \sum_{i=1}^{p} \frac{(\beta_i, \alpha)^2}{||\alpha||^2 ||\beta_i||^2} = 4\left(1 - \frac{(\gamma, \alpha)}{||\alpha||^2 ||\gamma||^2}\right) < 4\,,$$

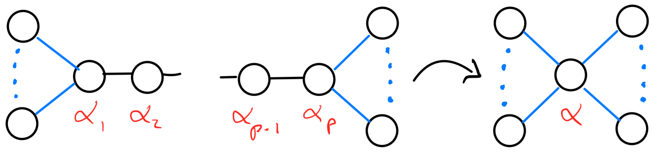
where the last inequality follows since $(\alpha, \gamma) \neq 0$. (Otherwise $\alpha \in \text{Span}(\beta_1, \ldots, \beta_p)$ which contradicts linear independence.) In conclusion, we have $\sum_{i=1}^{p} l_{\alpha\beta_i} < 4$ which is the desired statement. $\square$

Remark 6.2. *Lemma 6.4 excludes diagrams with four (or more) lines connected to one node such as*

Lemma 6.5. *If a chain with single-line links in an admissible diagram is replaced by a single node (attaching lines from other nodes to ends of the chain to the single, replacing node) the resulting diagram is admissible.*

Proof. Say that the nodes in the chain correspond to vectors $\alpha_1, \ldots, \alpha_p$ which are then replaced by a single node with associated vector α. Schematically, the replacement process changes the diagram as follows.



We have to show that the new diagram on the right is admissible given that the original one is, so we have to show that the new diagram satisfies Eq. (6.6) for a suitable definition of the vector α. If we define $\alpha = \sum_{i=1}^{p} \frac{\alpha_i}{||\alpha_i||}$, then a calculation similar to the one in Lemma 6.3 implies that

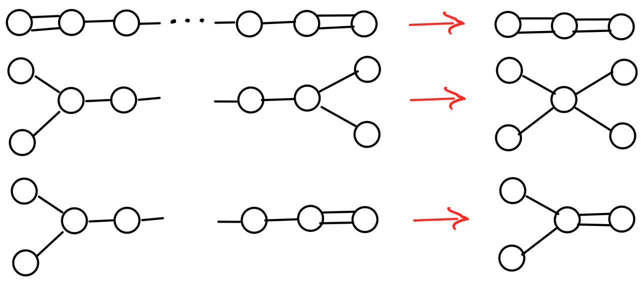
$$||\alpha||^2 = p - \sum_{i,j \text{ linked}} \sqrt{l_{\alpha_i \alpha_j}} = p - (p-1) = 1 \, .$$

Suppose β is a vector for one of the nodes not contained in the single-line chain. For its angle with the vector α we find

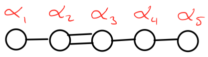
$$\cos\theta_{\beta\alpha} = \frac{(\beta,\alpha)}{||\beta||\,||\alpha||} = \sum_{i=1}^{p} \frac{(\beta,\alpha_i)}{||\beta||\,||\alpha_i||} = \left\{ \begin{array}{ll} \cos\theta_{\beta\alpha_1} = -\frac{\sqrt{l_{\beta\alpha_1}}}{2} & \text{if } \beta \text{ connected to } \alpha_1 \\ \cos\theta_{\beta\alpha_p} = -\frac{\sqrt{l_{\beta\alpha_p}}}{2} & \text{if } \beta \text{ connected to } \alpha_p \end{array} \right\} = -\frac{\sqrt{l_{\beta\alpha}}}{2} \, .$$

The last equality follows since the lines from β to either α_1 or α_p are preserved under the contraction and attached to α. Equality of the left- and right-hand side shows that the contracted diagram is admissible. $\square$

Remark 6.3. *The operation described in Lemma 6.5 excludes the following examples.*



The contracted diagrams on the right are excluded from Lemma 6.4 since they have four lines attached to the node in the middle. Therefore, from Lemma 6.5, the diagrams on the left must also be excluded. More generally, any chain with single line connections which has more than a total of three lines attached to the two nodes at its ends is excluded.

Lemma 6.6. *The diagram* 

is not admissible.

Proof. The proof is not exactly intuitive and relies on defining the two vectors

$$v = c_1 \frac{\alpha_1}{||\alpha_1||} + c_2 \frac{\alpha_2}{||\alpha_2||} \,, \qquad w = c_3 \frac{\alpha_3}{||\alpha_3||} + c_4 \frac{\alpha_4}{||\alpha_|} + c_5 \frac{\alpha_5}{||\alpha_5||} \,,$$

(corresponding to the two "arms" of the diagram at either side of the double line) and choosing the coefficients c_i such that v and w violate the Cauchy-Schwarz inequality. The quantities which enter this inequality are

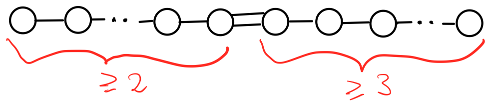
$$||v||^2 = c_1^2 + c_2^2 - c_1 c_2 \,, \qquad ||w||^2 = c_3^2 + c_4^2 + c_5^2 - c_3 c_4 - c_4 c_5 \,, \qquad (v, w) = -c_2 c_3 / \sqrt{2} \,.$$

These results follow by remembering that the number of lines in the diagram under consideration equals $-\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ for $||\alpha_i|| \geq ||\alpha_j||$. If we choose $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_4 = 2$ and $c_5 = 1$ (don't ask!) then

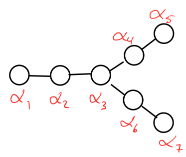
$$(v, w)^2 = 18 \,, \qquad ||v||^2 = 3 \,, \qquad ||w||^2 = 6 \,,$$

which violates $(v, w)^2 < ||v||^2 ||w||^2$. □

Remark 6.4. *We conclude that all diagrams of the form*



with a double line and single line arms with lengths at least 2 and 3 are not admissible since they can be contracted, from Lemma 6.5, to the diagram ruled out by Lemma 6.6.

Lemma 6.7. *The diagram* 

is not admissible.

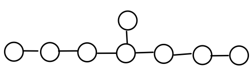
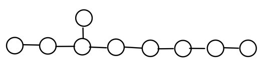
Proof. Define the three vectors

$$u = \frac{1}{\sqrt{3}} \left(2 \frac{\alpha_2}{||\alpha_2||} + \frac{\alpha_1}{||\alpha_1||} \right) \,, \quad v = \frac{1}{\sqrt{3}} \left(2 \frac{\alpha_4}{||\alpha_4||} + \frac{\alpha_5}{||\alpha_5||} \right) \,, \quad w = \frac{1}{\sqrt{3}} \left(2 \frac{\alpha_6}{||\alpha_6||} + \frac{\alpha_7}{||\alpha_7||} \right) \,,$$

which correspond to the three "arms" of the diagram. As in the proof of Lemma 6.4 it follows from $\alpha_3 \notin \mathrm{Span}(u, v, w)$ that

$$1 > \frac{(\alpha_3, u)^2}{||\alpha_3||^2 ||u||^2} + \frac{(\alpha_3, v)^2}{||\alpha_3||^2 ||v||^2} + \frac{(\alpha_3, w)^2}{||\alpha_3||^2 ||w||^2}$$

However, given that α_3 is only connected to α_2, α_4 and α_6 the expression on the right-hand side evaluated to 1, which is a contradiction. □

Lemma 6.8. *The diagrams*  ,  *are not admissible.*

Proof. This can be shown similarly to Lemma 6.7. For details see Ref. [1]. □

Classification of simple Lie algebras

We are now ready to classify simple Lie algebras by writing down all Dynkin diagrams which are consistent with the statements in Lemma 6.2 – Lemma 6.8. All these statements are consistent with the diagrams for A_n, B_n and D_n which we have already shown correspond to actual Lie algebras. In addition, there is a fourth infinite series, C_n, which corresponds to the **symplectic groups**, $\mathrm{Sp}(2n)$ which we have not considered explicitly. Apart from these four classical series, our rules only allow for five additional diagrams which correspond to the five **exceptional algebras**. All this is summarised in Table 6.3. Since our classification was

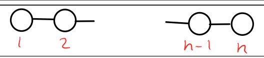
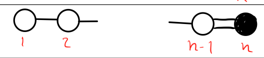
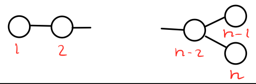
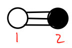
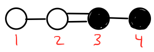
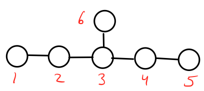
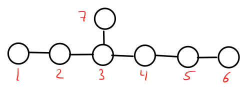
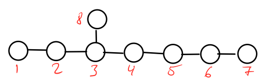
Dynkin diagram	$\mathcal{L}_\mathbb{C}$	$\mathcal{L}$	Cartan matrix A
	A_n	$\mathrm{su}(p,q),\ p+q=n+1$	Eq. (5.80)
	B_n	$\mathrm{so}(p,q),\ p+q=2n+1$	Eq. (5.159)
	C_n	$\mathrm{sp}(2n)$	$A(C_n)=A(B_n)^T$
	D_n	$\mathrm{so}(p,q),\ p+q=2n$	Eq. (5.154)
	G_2	G_2	$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$
	F_4	F_4	$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$
	E_6	E_6	$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$
	E_7	E_7	$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$
	E_8	E_8	$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

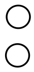
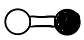
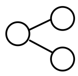
Table 6.3: List of all simple, complex Lie algbras, their Dynkin diagrams, some of their real sub-algebras and their Cartan matrices.

based on a necessary criterion we have to show, in principle, that all these diagrams do indeed correspond to simple, complex Lie algebras. We have done this explicitly for the A_n, B_n and D_n series. The C_n algebras follow from analysing the Lie algebras of the symplectic groups. Finally, it turns out that the five exceptional cases also correspond to actual algebras (and underlying groups) although we do not attempt an explicit construction here.

Another step is the classification of all real algebras (also called **real forms**) associated to the complexified algebras $\mathcal{L}$, which is a further part of the classification programme. We have done some of this explicitly in the context of examples. For instance, we have seen that the complex Lie algebras A_n contains all real Lie algebras $\mathrm{su}(p, q)$, where $p + q = n + 1$. Some of these real forms are indicated in column three of the above table but we will not pursue this systematically.

Relations between low-rank algebras

The Lie algebra is determined by the Dynkin diagram - as we will see soon the full set of roots can be reconstructed from the simple roots whose properties are encoded by the Dynkin diagram. Some entries in the above table degenerate at low rank and Dynkin diagrams which generically belong to a different series become identical. This indicates isomorphisms between specific low-rank algebras and some examples are listed in the following table.

$\mathrm{so}(3)_{\mathbb{C}}$	$\cong$	B_1	$\cong$	 $\cong$ 	$\cong$	A_1	$\cong$	$\mathrm{su}(2)_{\mathbb{C}}$	
$\mathrm{so}(4)_{\mathbb{C}}$	$\cong$	D_2	$\cong$		$\cong$	$A_1 \oplus A_1$	$\cong$	$\mathrm{su}(2)_{\mathbb{C}} \oplus \mathrm{su}(2)_{\mathbb{C}}$	
$\mathrm{so}(5)_{\mathbb{C}}$	$\cong$	B_2	$\cong$		$\cong$	C_2	$\cong$	$\mathrm{sp}(4)_{\mathbb{C}}$	(6.7)
$\mathrm{so}(6)_{\mathbb{C}}$	$\cong$	D_3	$\cong$		$\cong$	A_3	$\cong$	$\mathrm{su}(4)_{\mathbb{C}}$	

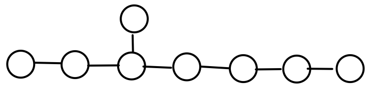
The final example, $\mathrm{so}(6)_{\mathbb{C}} \cong \mathrm{su}(4)_{\mathbb{C}}$, might be somewhat infuriating given how easy the equivalence follows here and how tricky it is to find the explicit isomorphism of the two matrix algebras.

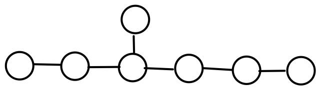
We have found four infinite series and 5 exceptional examples. Why can't we extend those exceptional algebras by adding nodes so that they form infinite series as well? At some level the answer is of course that Lemmas 6.2 – 6.8 exclude such a possibility. Adding more nodes to the G_2, the F_4 or the E_8 diagram leads to inadmissible diagrams. However, some of these higher exceptional algebras do exist as infinite-dimensional algebras, so called **Kac-Moody algebras**. For more information on this see, for example, Ref. [9].

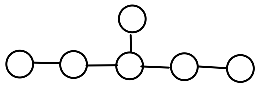
Another question is why some of the exceptional algebras with low ranks appear to be missing. The simple answer is that these diagrams degenerate and are already part of one of the infinite series. For example, the algebra one would call G_1 has a single dot Dynkin diagram so is, in fact, A_1. Likewise, the diagrams for what should be called F_n, with $n < 4$, obtained by removing dots from the F_4 diagram, are diagrams already contained in the A, B or C series. For exceptional E groups the story is more interesting.

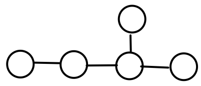
Application 6.1: *(Exceptional groups and unification)*

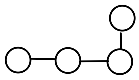
If we continue the E_n series down to $n < 6$ by successively removing dots from the right-hand side of the E_8 diagram we obtain the following.

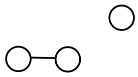
$E_8 \cong$ 

$E_7 \cong$ 

$E_6 \cong$ 

$E_5 \cong$  $\cong \quad D_5 \quad \cong \quad \mathrm{so}(10)_{\mathbb{C}}$ (6.8)

$E_4 \cong$  $\cong \quad A_4 \quad \cong \quad \mathrm{su}(5)_{\mathbb{C}}$

$E_3 \cong$  $\cong \quad A_2 \oplus A_1 \quad \cong \quad \mathrm{su}(3)_{\mathbb{C}} \oplus \mathrm{su}(2)_{\mathbb{C}}$

As for the other exceptional algebras, the lower rank algebras E_n with $n < 6$ are indeed already contained in the other series. However, the above list does show an interesting pattern, with E_3 the standard model gauge group (the $U(1)$ factor does not appear since we are considering semi-simple algebras), E_4 and E_5 the two favourite grand unification gauge groups and E_8 a prominent gauge group in string theory. Perhaps this suggests that exceptional E groups play an important role in physics - but of course this is speculation at this point.

6.2 Representations and Dynkin formalism

We have already discussed how representations can be described by weights. We would now like to approach this more systematically and work out how to construct weight systems of irreps, starting with the highest weight. We will generally be working with a semi-simple complex Lie algebra $\mathcal{L}$ of rank n with positive simple roots $(\alpha_1, \dots, \alpha_n)$ and study irreps $r : \mathcal{L} \to \mathrm{End}(V)$.

Structure of weight systems

We start by generalising the definition of α-strings to weights.

Definition 6.3. *For a root α of $\mathcal{L}$ and a weight w of r the **α-string through** w is the (maximal) subset of $\{w + k\alpha \,|\, k \in \mathbb{Z}\}$ which consists of weights of r. It is of the form*

$$w - p\alpha,\ w - (p-1)\alpha,\ \dots\ , w - \alpha,\ w,\ w + \alpha,\ \dots\ , w + (q-1)\alpha,\ w + q\alpha\ . \tag{6.9}$$

For these strings we have the analogue of Lemma 6.1.

Lemma 6.9. *For a root $\alpha \in \Delta$, let $w \in \mathcal{H}'$ be a weight of r such that $w + \alpha$ is not a weight of r. Then, the α-string through w has the form $w, w - \alpha, \dots, w - p\alpha$, where*

$$p = \frac{2(w, \alpha)}{(\alpha, \alpha)}\ . \tag{6.10}$$

Proof. The proof is similar to the proof of Lemma 6.1. We focus on the $\mathrm{su}(2)_{\mathbb{C}}$ subalgebra of $\mathcal{L}$ spanned by $(H_\alpha, E_\alpha, E_{-\alpha})$ under which the states $v_w, \ldots, v_{w-p\alpha} \in V$ with weights $w, \ldots, w-p\alpha$ must form a representation with a certain spin j, such that the state v_w corresponds to eigenvalue $-j$ and $v_{w-p\alpha}$ corresponds to $+j$. Since

$$r(H_\alpha)(v_w) = (w, \alpha)v_w \ , \qquad r(H_\alpha)(v_{w-p\alpha}) = (w - p\alpha, \alpha)v_{w-p\alpha} \ ,$$

we have $(w, \alpha) = -(\alpha, \alpha)j$ and $(w - p\alpha, \alpha) = (\alpha, \alpha)j$ and adding these equations immediately leads to Eq. (6.10). $\square$

The point of this lemma is that it tells us how many times we can take away a root from a given weight and still obtain weights of the representation in question. Knowing this is key to the algorithm which generates the entire weight system of a representation. This algorithm needs a bit of bookkeeping which is what the next definition is about.

Definition 6.4. *Let λ be the highest weight of r and $w = \lambda - \sum_{i=1}^n m_i\alpha_i$ a weight of r. The sum $\sum_{i=1}^n m_i$ is called the **level** of the weight w.*

So the level equals the number of times we have to subtract simple roots from the highest weight in order to arrive at the weight in question. We are now ready to formulate the algorithm for computing weight systems.

Algorithm *(Computing weight systems of irreps)*

(1) Choose an irrep by choosing a highest weight λ (a Dynkin label with all entries ≥ 0). This fixes the weight at level 0.

(2) Assume that all weights up to level s have been constructed and proceed as follows.
(a) Find all weights w at level s and all α_i such that $w + \alpha_i$ is not a weight. (Since $w + \alpha_i$ is at level $s - 1$ this can be decided.)
(b) For w and α_i as in (a) add the weights $w - \alpha_i, \ldots, w - a_i\alpha_i$ to the list of weights, where $a_i = \frac{2(w,\alpha_i)}{(\alpha_i,\alpha_i)}$.

(3) Iterate the process until no more new weights are found.

The reason this algorithm works is of course Eq. (6.10). It is particularly convenient to carry out if all roots and weights are represented by Dynkin labels. The Dynkin labels of the positive simple roots α_i are simply the rows of the Cartan matrix A of $\mathcal{L}$. If the weights w are represented by Dynkin labels $(a_1, \ldots, a_n)$ then the entry a_i of the Dynkin label equals the number of times α_i can be subtracted from w, in accordance with step (2b) of the algorithm.

Examples

We would like to practice the above algorithm with a few examples, starting with the simplest case, $\mathcal{L} = A_1$.

Weights of A_1 representations

The Cartan matrix of $A_1 \cong \mathrm{su}(2)_{\mathbb{C}}$ is $A(A_1) = (2)$. Suppose we consider an irrep with highest weight Dynkin label (a), where $a \geq 0$. Then, according to our algorithm, the single root (2) can be subtracted from (a) precisely a times and this leads to the weight system

$$(a), \ (a-2), \ \ldots, (-a+2), \ (-a) \ .$$

The dimension of the representation is $\dim(r) = a + 1$ and, by comparing dimensions, we conclude that $a = 2j$, where $j \in \mathbb{Z}/2$ is the spin normally used in physics to label $\mathrm{su}(2)_{\mathbb{C}}$ representations.

Weights of A_2 representations

The Cartan matrix of $A_2 \cong \mathrm{su}(3)_{\mathbb{C}}$ is given by

$$A(A_2) = \left(\begin{array}{rr} 2 & -1 \\ -1 & 2 \end{array} \right) \,,$$

and the Dynkin label of the two simple roots are the two rows $\alpha_1 \sim (2, -1)$ and $\alpha_2 \sim (-1, 2)$ of $A(A_2)$. For the fundamental and complex conjugate fundamental with highest weight Dynkin label $(1, 0)$ and $(0, 1)$, respectively, we find the weight systems

$$\mathbf{3}: \quad \underbrace{(1,0)}_{u} \overset{\alpha_1}{\to} \underbrace{(-1,1)}_{d} \overset{\alpha_2}{\to} \underbrace{(0,-1)}_{s} \,, \qquad \bar{\mathbf{3}}: \quad \underbrace{(0,1)}_{\bar{s}} \overset{\alpha_2}{\to} \underbrace{(1,-1)}_{\bar{d}} \overset{\alpha_1}{\to} \underbrace{(-1,0)}_{\bar{u}} \,. \tag{6.11}$$

(We have indicated underneath, in quark notation, which vectors these weights correspond to.) For the two-index symmetric tensor representation **6** with highest weight Dynkin label $(2, 0)$ we find

$$\mathbf{6}: \quad (2,0) \overset{\alpha_1}{\to} (0,1) \begin{array}{c} \overset{\alpha_2}{\nearrow} (1,-1) \overset{\alpha_1}{\searrow} \\ \underset{\alpha_1}{\searrow} (-2,2) \underset{\alpha_2}{\nearrow} \end{array} (-1,0) \overset{\alpha_2}{\to} (0,-2) \,. \tag{6.12}$$

For the adjoint, **8**, with highest weight Dynkin label $(1, 1)$ we have

$$\mathbf{8}: \quad (1,1) \begin{array}{c} \overset{\alpha_2}{\nearrow} (2,-1) \overset{\alpha_1}{\to} (0,0) \overset{\alpha_1}{\to} (-2,1) \overset{\alpha_2}{\searrow} \\ \underset{\alpha_1}{\searrow} (-1,2) \overset{\alpha_2}{\to} (0,0) \overset{\alpha_2}{\to} (1,-2) \underset{\alpha_1}{\nearrow} \end{array} (-1,-1) \tag{6.13}$$

Tensor products - again

Once the weight systems of irreps are know it is straightforward (although tedious for larger representations) to work out the weight systems of tensor products and their Clebsch-Gordan decomposition. As a simple example, consider the A_2 tensor product $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$. Forming all sums of the $\mathbf{3}$ and $\bar{\mathbf{3}}$ weights in Eq. (6.11) gives the weights of the 9 tensor states

$$(1,1),\ (2,-1),\ (0,0),\ (-1,2),\ (0,0),\ (-2,1),\ (0,0),\ (1,-2),\ (-1,-1) \,. \tag{6.14}$$

The highest weight in this representation is $(1, 1)$ since neither $(1, 1) + \alpha_1$ nor $(1, 1) + \alpha_2$ appears in this list. We immediately conclude that the $\mathbf{3} \times \bar{\mathbf{3}}$ must contain the irrep with highest weight Dynkin label $(1, 1)$, that is, the adjoint **8**. From the above algorithm, we can now generate all the weights of the **8** representation, as we have done in Eq. (6.13), and these should be taken away from the list (6.14). These steps can then be repeated for the remaining weights, in the present case just a single weight $(0, 0)$ which, of course, corresponds to the trivial representation **1**.

Dynkin and dual basis

A weight w can be represented by a Dynkin label $(a_1, \ldots, a_n)$ or by a dual vector $(\bar{w}_1, \ldots, \bar{w}_n)$ with components each defined by

$$a_i = \frac{2(w, \alpha_i)}{(\alpha_i, \alpha_i)} \, , \qquad w = \sum_{i=1}^{n} \bar{w}_i \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \, . \tag{6.15}$$

So the dual vector is just the coordinate vector of the weight relative to the basis of simple positive roots. The factor $2/(\alpha_i, \alpha_i)$ has been included for convenience, and is based on the convention that the longer roots in a Dynkin diagram are normalised to length 2. It is straightforward to find the relation between the Dynkin labels and the dual vector, by inserting the second Eq. (6.15) into the first.

$$a_i = \sum_j \bar{w}_j \frac{2A_{ji}}{(\alpha_j, \alpha_j)} \qquad \Rightarrow \qquad \bar{w}_j = \sum_i a_i G_{ij} \quad \text{where} \quad G_{ij} = \frac{(\alpha_j, \alpha_j)}{2} (A^{-1})_{ij} \, . \tag{6.16}$$

The matrix G is also called the **metric tensor** and it can be used to express the Killing form of two weights w, w',

$$(w, w') = \sum_{i,j} \frac{2\bar{w}_i}{(\alpha_i, \alpha_i)} (\alpha_i, \alpha_j) \frac{2\bar{w}'_j}{(\alpha_j, \alpha_j)} = \sum_{i,j} \underbrace{\frac{2\bar{w}_i}{(\alpha_i, \alpha_i)} A_{ij}}_{\to a_j} \bar{w}'_j = \sum_i a_i \bar{w}'_i = \sum_{i,j} a'_i G_{ij} a_j \, , \tag{6.17}$$

in terms of their Dynkin labels. The metric tensors can be explicitly worked out from the Cartan matrices, basically inverting it, and the complete list of metric tensors can be found in Ref. [4]. For example, for A_2 the metric tensor is given by

$$G(A_2) = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \, . \tag{6.18}$$

Charges

By a charge, Q, we mean an element of the Cartan $\mathcal{H}$, so a generator of a U(1). When a representation r is given in terms of a list of weights w it is often desirable to compute the charge values $w(Q)$ of these weights and this can be done easily using Dynkin labels. A convenient way to represent the charge Q is relative to the basis (H_{α_i}) of the Cartan and write

$$Q = \sum_i \frac{2\bar{q}_i}{(\alpha_i, \alpha_i)} H_{\alpha_i} \, , \qquad w = \sum_j \bar{w}_j \frac{2\alpha_j}{(\alpha_j, \alpha_j)}$$

so that the charge Q is represented by a vector $(\bar{q}_1, \ldots, \bar{q}_n)$. A short calculation then shows that the charge of the state with weight w can be computed as

$$w(Q) = \sum_{i,j} \frac{2\bar{q}_i}{(\alpha_i, \alpha_i)} \frac{2\bar{w}_j}{(\alpha_j, \alpha_j)} \underbrace{\alpha_j(H_{\alpha_i})}_{=(\alpha_j, \alpha_i)} = \sum_{i,j} \bar{q}_i \bar{w}_j \frac{2A_{ji}}{(\alpha_j, \alpha_j)} = \sum_i \bar{q}_i a_i \, . \tag{6.19}$$

In conclusion, the charge $w(Q)$ of the weight w can simply be computed by a dot product between the dual basis vector $(\bar{q}_1, \ldots, \bar{q}_n)$ which represents the charge and the Dynkin label of the weight w.

Application 6.2: *(Charges in the quark model)*

To illustrate how the above formalism for charges works in practice, let us determine some charges in the SU(3) quark model. Suppose we are interested in isospin, T_3, and electrical charge Q. First we have to determine the associated dual basis vectors $\bar{T}_3$ and $\bar{Q}$ which we can just dot into Dynkin labels to obtain charge values, as in Eq. (6.19). This can be done in various ways but one quick method is to use two known values of the charges, for example, the u-quark has isopsin $1/2$ and electrical charge $2/3$ and and $\bar{s}$-quark isospin 0 and electrical charge $1/3$. (These conditions might be viewed as a definition of what we mean by isospin and electrical charge in this context.) Comparing with Eq. (6.11) this translates into the constraints

$$\begin{array}{rclcrclcrcl} \bar{T}_3\cdot(1,0) &=& \frac{1}{2}\,, && \bar{T}_3\cdot(0,1) &=& 0 && \Rightarrow && \bar{T}_3 &=& \frac{1}{2}(1,0) \\ \bar{Q}\cdot(1,0) &=& \frac{2}{3}\,, && \bar{Q}\cdot(0,1) &=& \frac{1}{3} && \Rightarrow && \bar{Q} &=& \frac{1}{3}(2,1) \end{array}\,.$$

The point is that the so-determined charge vectors now work for all weights in all representations. For example, consider the weight $(-1,2)$ in the **8** representation in Eq. (6.13). Its charges are

$$\bar{T}_3\cdot(-1,2) = -\frac{1}{2}\,, \qquad \bar{Q}\cdot(-1,2) = 0$$

which identifies $(-1,2)$ as the weight of a state with isospin $-1/2$ and electrical charge 0, that is, as a Kaon K^0.

Dimensions and degeneracies

So far we have glossed over a subtlety: weights can be degenerate, that is, the associated eigenspaces may have dimensions larger than one. If such degenerate weights are present in a representation, simply counting the weights does not give us the dimension of the representation. In fact, we already know that this happens. For the adjoint representation, all non-zero weights (= roots) are non-degenerate while the zero weight (which corresponds to the Cartan) has degeneracy equal to the rank of the algebra. This is evident in the weight system (6.13) of the adjoint **8** of A_2, where we find 6 non-zero weights, and the weight zero has degeneracy two. While we know the degeneracies for the adjoint the same is not true for all other representations, so we need additional information. Also note that the degeneracy does *not*, in general, equal the number of times a certain weight is produced by our algorithm. All the algorithm does is produce a list of the distinct weights - their degeneracies still have to be determined. This can be done with the **Freudenthal formula**.

Theorem 6.2. *(Freudenthal formula) For an irrep with highest weight λ the degeneracies g_w of weights w are given by the recursion formula*

$$[(\lambda+\delta,\lambda+\delta)-(w+\delta,w+\delta)]\,g_w = 2\sum_{\alpha\in\Delta_+,k\geq 0} g_{w+k\alpha}(w+k\alpha,\alpha)\,, \tag{6.20}$$

where $\delta = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ and the Dynkin label of δ is $(1,1,\ldots,1,1)$.

Proof. For the proof see Ref. [1], 25.1. □

The Freudenthal formula allows for a recursive calculation of degeneracies, starting with the highest weight λ which we know has degeneracy $g_\lambda = 1$ and then proceeding level by level. It is also useful to have a formula for the dimension of a representation in terms of the highest weight, the **Weyl formula**.

Theorem 6.3. *(Weyl formula) The dimension* $\dim(\lambda)$ *of an irrep with highest weight* λ *is given by*

$$\dim(\lambda) = \prod_{\alpha\in\Delta_+} \frac{(\lambda+\delta,\alpha)}{(\delta,\alpha)} \tag{6.21}$$

where δ *is as defined in Theorem* (6.2).

Proof. See Ref. [1], 24.1. □

If the dimension calculated from this formula coincides with the number of different weights then it is clear that all weights must be non-degenerate, so in this case there is no need to use the Freudenthal formula. There is also a related and useful formula for the value of the quadratic Casimir.

Theorem 6.4. *The value* $C(\lambda)$ *of the quadratic Casimir for an irrep with highest weight* λ *is given by* $C(\lambda) = (\lambda, \lambda + 2\delta)$.

Proof. The quadratic Casimir can be written in terms of the Cartan-Weyl basis as

$$C = \gamma^{IJ} T_I T_J = \sum_{i,j} \gamma^{ij} H_{\alpha_i} H_{\alpha_j} + \sum_{\alpha\in\Delta} E_\alpha E_{-\alpha}$$

where we have been using the result Eq. (4.27) for the Killing form and $\gamma_{ij} = \Gamma(H_{\alpha_i}, H_{\alpha_j}) = (\alpha_i, \alpha_j) = A_{ij}\frac{(\alpha_j,\alpha_j)}{2}$. Inserting the inverse, γ^{ij}, into the above formula for C gives

$$C = \sum_{i,j} G_{ij} \frac{2H_{\alpha_i}}{(\alpha_i,\alpha_i)} \frac{2H_{\alpha_j}}{(\alpha_j,\alpha_j)} + \sum_{\alpha\in\Delta} E_\alpha E_{-\alpha}$$

To find $C(\lambda)$ it is enough to evaluate C on the highest weight vector v (it takes the same value on the entire representation vector space) with highest weight λ and highest weight Dynkin label $a_i = \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}$. This leads to

$$C(v) = \sum_{i,j} G_{ij} a_i a_j v + \sum_{\alpha\in\Delta_+} \underbrace{[E_\alpha, E_{-\alpha}]}_{=H_\alpha}(v) = \left[(\lambda,\lambda) + \sum_{\alpha\in\Delta_+} (\lambda,\alpha)\right] v = (\lambda, \lambda+2\delta) v\ .$$

□

Example: dimensions and Casimir for A_2

The positive roots (written as Dynkin labels) of A_2 are

$$\Delta_+ = \{\alpha_1 = (2,-1)^T, \alpha_2 = (-1,2)^T, \beta = \alpha_1 + \alpha_2 = (1,1)^T\}\ ,$$

the metric tensor G has already been given in Eq. (6.18) and $\delta = (1,1)^T$. It follows that

$$G\alpha_1 = (1,0)^T\ , \quad G\alpha_2 = (0,1)^T\ , \quad G\beta = (1,1)^T\ ,$$

and inserting all this into the Weyl formula for an irrep with highest weight Dynkin label (a_1, a_2) gives

$$\begin{aligned}\dim(a_1, a_2) &= \frac{(a_1+1, a_2+1)G\alpha_1}{\delta^T G\alpha_1}\frac{(a_1+1, a_2+1)G\alpha_2}{\delta^T G\alpha_2}\frac{(a_1+1, a_2+1)G\beta}{\delta^T G\beta} \\ &= \frac{1}{2}(a_1+a_2+2)(a_1+1)(a_2+1) \end{aligned} \tag{6.22}$$

As a sanity check we insert $(a_1, a_2) = (1, 1)$ for the adjoint and find $\dim(1, 1) = 8$, as expected. To have a closed formula for the dimensions of all A_2 representations is not a minor accomplishment.

For the Casimir we have from Theorem 6.4

$$C(a_1, a_2) = (a_1, a_2)\, G \begin{pmatrix} a_1+2 \\ a_2+2 \end{pmatrix} = \frac{2}{3}(a_1^2 + a_2^2 + a_1a_2 + 3a_1 + 3a_2)\,, \tag{6.23}$$

and combining the above result, the index, Eq. (4.13), can be written as

$$\begin{aligned} c(a_1, a_2) &= \frac{\dim(a_1, a_2)}{\dim(1, 1)}\, C(a_1, a_2) \\ &= \frac{1}{24}(a_1+a_2+2)(a_1+1)(a_2+1)(a_1^2 + a_2^2 + a_1a_2 + 3a_1 + 3a_2)\,. \end{aligned} \tag{6.24}$$

For the index of the fundamental and complex conjugate fundamental this gives $c(1, 0) = c(0, 1) = 1$ while the index of the adjoint is $c(1, 1) = 6$.

6.3 Subgroups and branching

Can subalgebras and the branching of representations be described in terms of the Dynkin formalism of roots and weights? Unfortunately, this does not work in all cases and depends on whether a subalgebra is **regular**, that is, whether its Cartan-Weyl decomposition is consistent with the Cartan-Weyl decomposition of the original algebra. As we will see, not all subalgebras are regular in this sense and for such cases special considerations are required. However, many subalgebras are regular and for all these cases a uniform formalism can be developed. We will now explain how this works.

Regular subalgebras

We call a subalgebra $\mathcal{L}' \subset \mathcal{L}$ of a semi-simple complex Lie algebra $\mathcal{L}$ **maximal** if a subalgebra $\tilde{\mathcal{L}}$ with $\mathcal{L}' \subsetneqq \tilde{\mathcal{L}} \subsetneqq \mathcal{L}$ does not exist. We will be interested in understanding such maximal subalgebras - smaller algebras can be obtained by iterating the process and applying it to the maximal subalgebras and so forth.

If we want to describe subalgebras in terms of the Dynkin formalism we require that Cartan-Weyl decompositions of algebra and subalgebra are compatible, for example if $\mathcal{L}' \subset \mathcal{L}$ then the Cartan of the subalgebra should be contained in the Cartan of the algebra, so $\mathcal{H}' \subset \mathcal{H}$. To see that this is not automatic, consider the obvious group embedding $SO(3) \subset SU(3)$, induced by the fact that orthogonal matrices are also unitary. The Cartan of $SO(3)$ is one-dimensional and generated, for example, by the off-diagonal matrix σ_{12} in Eq. (5.140). The Cartan of $SU(3)$, on the other hand, consists of diagonal, traceless matrices and this set clearly does not contain σ_{12}. The following discussion excludes such "irregular" cases and focuses on regular subalgebras defined as follows.

Definition 6.5. *A maximal algebra $\mathcal{L}' \subset \mathcal{L}$ of $\mathcal{L}$ is called* ***regular*** *if $\mathcal{H}' \subset \mathcal{H}$ and $\mathrm{Span}(E'_\alpha) \subset \mathrm{Span}(E_\alpha)$.*

The following examples show that regular subalgebras come in (at least) two types. First, consider the subgroup $\mathrm{SU}(n-1) \times \mathrm{U}(1) \subset \mathrm{SU}(n)$ with embedding

$$\mathrm{SU}(n-1) \ni U \mapsto \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SU}(n) \;, \qquad \mathrm{U}(1) \ni z \mapsto \mathrm{diag}(z, \dots, z, z^{-n+1}) \in \mathrm{SU}(n) \;.$$

The Cartan-Weyl compatibility is apparent and maximality can be shown from commutation relations, so the embedding is regular. This means we can have regular subalgebras of the type $\mathcal{L}' \oplus \mathrm{u}(1)_{\mathbb{C}} \subset \mathcal{L}$, which consist of a semi-simple algebra $\mathcal{L}'$ one rank lower than $\mathcal{L}$ and a $\mathrm{u}(1)$ part.

As an example of the other type, consider the embedding $\mathrm{SO}(2n) \subset \mathrm{SO}(2n+1)$ defined by

$$\mathrm{SO}(2n) \ni R \mapsto \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(2n+1) \;.$$

Maximality can be shown from the commutation relations and Cartan-Weyl compatibility follows from our discussion of the Lie algebras of orthogonal groups. In conclusion, we see that D_n is a regular subalgebra of B_n - so this is a case where the regular subalgebra is semi-simple, has the same rank as the original algebra and there is no $\mathrm{u}(1)$ part.

It turns out both types of regular subalgebras can be obtained from Dynkin diagrams. Roughly the idea is that removing a node from the Dynkin diagram for $\mathcal{L}$ will give the Dynkin diagram of a regular subalgebra $\mathcal{L}'$. As we will see, this is in fact precisely how it works for the first above type of regular subalgebras. For the second type we need to preserve the rank, while removing a node from the Dynkin diagram lowers the rank by one. This problem can be resolved by considering **extended Dynkin diagrams** which are the original Dynkin diagrams plus one additional node.

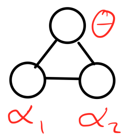
Extended Dynkin diagrams

If we add one additional node to a rank n Dynkin diagram for $\mathcal{L}$ with simple positive roots $(\alpha_1, \dots, \alpha_n)$ the resulting diagram is of course not admissible, since $n+1$ vectors cannot be linearly independent in $\mathcal{H}'$ which is of dimension n. However, we can try to ensure that the additional node is added such that subsequently removing one (any) node from the extended Dynkin diagram leads to a viable Dynkin diagram for a semi-simple Lie algebra. In view of Lemma 6.1 there is only one way to do this: add to the Dynkin diagram the node which corresponds to the lowest root θ, that is the root for which all $\theta - \alpha_i$ are not roots. The resulting diagram with $n+1$ nodes and corresponding roots $(\alpha_1, \dots, \alpha_n, \theta)$ is called the **extended Dynkin diagram** of $\mathcal{L}$. If any one of the roots $(\alpha_1, \dots, \alpha_n, \theta)$ of the extended diagram is removed we remain with a basis of $\mathcal{H}'$ and the resulting system is guaranteed to satisfy the requirements on simple positive roots stated in Lemma 6.1. This is not sufficient to guarantee that we always end up with viable Dynkin diagrams but this can be checked explicitly by constructing all extended Dynkin diagrams and then removing one node.

To see how this works let us construct the extended Dynkin diagram of A_2. The simple positive roots of A_2 (as Dynkin labels) are $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$ and inspection of the root system in Eq. (6.13) shows that $\theta = (-1, -1)$ is the lowest root. Hence, the extended Dynkin diagram corresponds to the roots $(\alpha_1, \alpha_2, \theta)$. To work out the diagram we need to check the lengths of these roots and the number of lines between them. Using the metric tensor (6.18) we find that $(\theta, \theta) = (\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$, so all three roots have the same length 2. Further $(\theta, \alpha_1) = (\theta, \alpha_2) = -1$ implies that

$$n_{\theta\alpha_i} = \frac{2(\theta, \alpha_i)}{(\alpha_i, \alpha_i)} = -1$$

for $i = 1, 2$ which means that θ should be connected to α_1 and α_2 with one line each. Hence, the resulting extended Dynkin diagram for A_2 is



The process just described for A_2 can be carried out for all algebras and associated Dynkin diagrams in our classification, Table 6.3. The resulting list of extended Dynkin diagrams is provided in Table 6.4. It can be checked by inspection that removing one node from any of these extended diagrams leads to a viable Dynkin diagram for a semi-simple algebra.

Finding regular subalgebras

Regular subalgebras can be obtained from the Dynkin diagram and the extended Dynkin diagram.

Theorem 6.5. *The regular subalgebras of $\mathcal{L}$ are*
(i) semi-simple algebras $\mathcal{L}'$ whose Dynkin diagram is found by removing one node (and the lines to it) from the extended Dynkin diagram of $\mathcal{L}$.
(ii) a sum $\mathcal{L}' \oplus \mathrm{u}(1)_{\mathbb{C}}$, where $\mathcal{L}'$ is a semi-simple algebra whose Dynkin diagram is obtained by removing one node (and the lines to it) from the Dynkin diagram of $\mathcal{L}$.

Proof. See, for example, Ref. [4]. □

An example for how to apply this theorem, for the case of E_8, is given in the Table 6.5.

Branching

For a regular subalgebra $\mathcal{L}' \subset \mathcal{L}$ the compatibility of the Cartan-Weyl decomposition means that any set of basis generators for the Cartan $\mathcal{H}'$ can be written as linear combinations of basis generators of $\mathcal{H}$. This means that weights of representations of $\mathcal{L}$ and the weights which emerge by branching to $\mathcal{L}'$ must be related linearly. In other words, if $n = \mathrm{rk}(\mathcal{L})$ and $n' = \mathrm{rk}(\mathcal{L}')$ there exists a $n' \times n$ **projection matrix** $P(\mathcal{L}' \subset \mathcal{L})$ which maps the Dynkin labels of weights in any $\mathcal{L}$ irrep r to the weights of the $\mathcal{L}'$ representation r' it branches to. This matrix only depends on the algebras and their embedding but not on the representation r. Once determined, for example by looking at the branching of simple low-dimensional irreps, it can be applied to all representations.

This is probably best illustrated by a simple example. Consider the isospin sub-algebra $\mathrm{su}_I(2)_{\mathbb{C}} \subset A_2 = \mathrm{su}(3)_{\mathbb{C}}$. In this case, the projection matrix is just a 1×2 matrix, mapping two-dimensional Dynkin labels of A_2 to one-dimensional Dynkin label of A_1. If we start

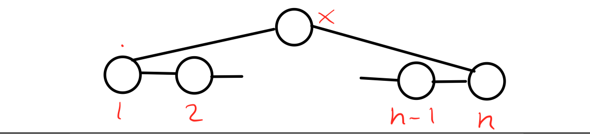
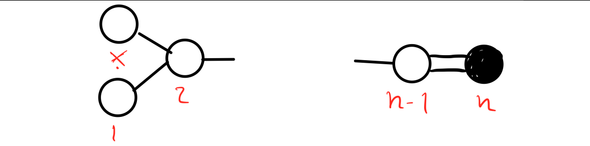
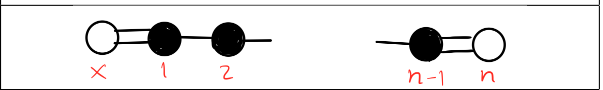
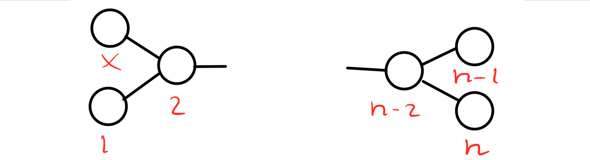
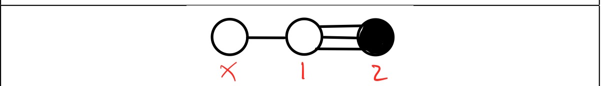
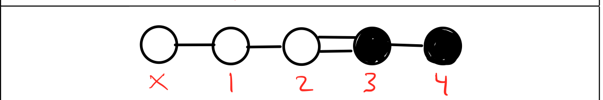
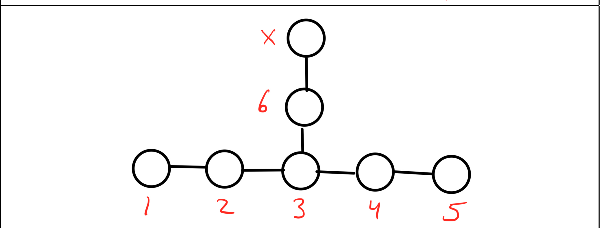
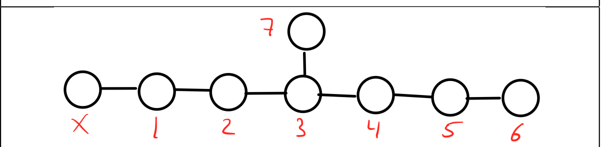
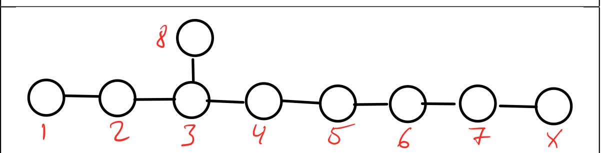
$\mathcal{L}$	extended Dynkin diagram
A_n	
B_n	
C_n	
D_n	
G_2	
F_4	
E_6	
E_7	
E_8	

Table 6.4: Extended Dynkin diagrams. The new node which corresponds to the lowest root θ is indicated by an x.

with $P = (p_1, p_2)$ and require that

$$P\overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{u} = 1 \,, \qquad P\overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{\bar{s}} = 0 \,,$$

which means, in quark model language, the u quark has isospin $1/2$ and the $\bar{s}$ quark has isospin 0 (this is one way to specify what we mean by the isospin subalgebra) we have

$$P(\mathrm{su}_I(2)_{\mathbb{C}} \subset \mathrm{su}(3)_{\mathbb{C}}) = (1, 0) \;. \tag{6.25}$$

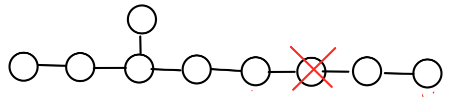
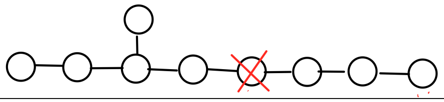
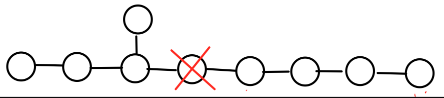
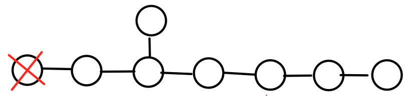
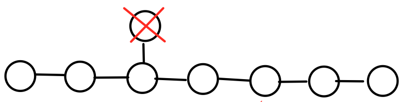
extended E_8 diagram	semi-simple subalgebra
	$E_6 \oplus A_2 \cong E_6 \oplus \mathrm{su}(3)_{\mathbb{C}}$
	$D_5 \oplus A_3 \cong \mathrm{so}(10)_{\mathbb{C}} \oplus \mathrm{su}(4)_{\mathbb{C}}$
	$A_4 \oplus A_4 \cong \mathrm{su}(5)_{\mathbb{C}} \oplus \mathrm{su}(5)_{\mathbb{C}}$
E_8 diagram	**non semi-simple subalgebra**
	$D_7 \oplus \mathrm{u}(1)_{\mathbb{C}} \cong \mathrm{so}(14)_{\mathbb{C}} \oplus \mathrm{u}(1)_{\mathbb{C}}$
	$A_7 \oplus \mathrm{u}(1)_{\mathbb{C}} \cong \mathrm{su}(8)_{\mathbb{C}} \oplus \mathrm{u}(1)_{\mathbb{C}}$

Table 6.5: Some regular semi-simple subgroups of E_8, obtained by deleting a node from the extended E_8 Dynkin diagram and some regular non semi-simple subgroups of E_8, obtained by deleting a node from the E_8 Dynkin diagram.

Applying this matrix, for example, to the weights of the $\mathbf{6}$ representation in Eq. (6.12) we find

$$\begin{array}{ccccccccc}
(2,0) & & (2) & & (2) & & & & \\
(0,1) & & (1) & & & & (1) & & \\
(-2,2),(1,-1) & \xrightarrow{P} & (0),(0) & = & (0) & \oplus & & \oplus & (0) \\
(-1,0) & & (-1) & & & & (-1) & & \\
(0,-2) & & (-2) & & (-2) & & & & \\
\mathbf{6} & \longrightarrow & & & \mathbf{3} & \oplus & \mathbf{2} & \oplus & \mathbf{1}
\end{array}$$

so the $\mathbf{6}$ of A_2 branches into an $\mathrm{su}(2)_{\mathbb{C}}$ triplet, a doublet and a singlet.

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