Simulation and modeling of nonlinear mirror modes

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Outline

- Satellite observations
- Vlasov-Maxwell simulations of the mirror instability and of its nonlinear developments: magnetic “holes” or “humps”
- Theoretical interpretations
- Bistability: Magnetic holes below instability threshold
- Conclusions and open questions
1. Satellite observations

Magnetic structures (humps or holes) that are quasi-stationary in the plasma frame, with no or little change in the magnetic field direction are commonly observed in the solar wind and the planetary magnetosheaths.

![Graph showing magnetic structures]

**Figure 1.** Each panel shows 3 hours of Galileo magnetometer field magnitude data (solid black line), appropriate quartiles (dotted), and the median value (solid gray) computed using 20 min sliding windows with single sample shifts. The panels show examples of “peaks” (top), “dips” (middle), and “other” (bottom) structures.


**Structures observed in the terrestrial magnetosheath**

![Cluster data showing magnetic structures]

**Figure 1.** An example of mirror mode structures of the two types. Top panel: peaks (peakness = 0.83), bottom panel: dips (peakness = -1.92).

Soucek, Lucek & Dandouras JGR **113**, A04203 (2008)

**Structures observed in the Jovian magnetosheath**

Usually viewed as nonlinear mirror modes
Mirror–mode structures in the wake of Io, as observed by Galileo

Russell et al., JGR 104 (A8) 17471 (1999)
Huddleston et al., JGR 104 (A8) 17479 (1999)
Mirror modes in Venus’ magnetosheath

Volwerk et al., GRL 35, L12204 (2008)
JGR 113, E00B16 (2008)

Figure 1. The magnetic field data for 5 May 2006. The spacecraft moves from the solar wind (SW), through the BS (BS) into the magnetosheath (MS). (e) and (f) The fluctuation of the magnetic field $\Delta B/B$ and the angle $\theta_{Bmv}$ between the maximum variance direction and the mean magnetic field (dots), and the angle $\beta_{Bmv}$ between the minimum variance direction and the mean magnetic field (pluses).

MM waves are identified as having large strengths $\Delta B/B$, and small angles $\theta_{Bmv}$ between the maximum variance and the magnetic field direction $\theta_{Bmv} \leq 30^\circ$ [Price et al., 1986]. (linear holes)

Figure 3. The magnetic field data for 2 October 2006. The spacecraft moves from periapsis in Venus’ magnetosphere (MSP), through the magnetopause (MP) into the magnetosheath (MS). (e) and (f) The fluctuation of the magnetic field $\Delta B/B$; the angle $\theta_{Bmv}$ between the maximum variance direction and the mean magnetic field (dots), and the angle $\beta_{Bmv}$ between the minimum variance direction and the mean magnetic field (pluses).
Conditions for peaks or dips

Figure 2. A, B and C show the field magnitude recorded during three intervals from 20 Dec 1997. They illustrate three forms of mirror structure: peaks, dips and a near sinusoidal waveform.

Lucek et al. GRL 26, 2159 (1999)

Structures observed in the terrestrial magnetosheath
Depending on local values of $\beta$, magnetic holes or humps are preferentially formed.

Same conclusion by Bavassano-Cattaneo et al. 1998 (Saturn’s magnetosheath),
Soucek, Lucek & Dandouras 2008 (Earth’s magnetosheath).
From Phan et al. JGR 99, 121 (1994)

anticorrelation magnetic field - density

magnetosheath region adjacent to the dayside magnetopause (AMPTE/IRM satellite).

pressure equilibrium
Strong anti-correlation of magnetic field and density for nearly sinusoidal (left) and peak (right) mirror modes.


Measurement by AMPTE-UKS satellite in the magnetosheath.
Magnetic holes may display different shapes (sharp or wide)

Free solar wind (Ulysses)

![Graph](image1)

Jovian magnetosheath (Ulysses)

![Graph](image2)

Free solar wind (Helios)

![Graph](image3)

**Figure 1.** Examples of magnetic holes observed (a) by Ulysses in the free solar wind (taken from Figure 2 of Winterhalter et al. [1994]), (b) by Ulysses in the magnetosheath of Jupiter, called mirror mode structures (from Figure 5 of Erdős and Balogh [1996]), and (c) by Helios in the free solar wind (data courtesy of K. Spence and F.M. Neubauer, University of Köln, 1999). Shown is the magnetic field magnitude.

Baumgärtel JGR 104 (A12), 28295 (1999)

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Winterhalter et al. (2000)

**Figure 2.** More examples of the mirror mode structures in the solar wind.

Zhang et al. GRL 35, L10106 (2008)

Do these structures have a unique origin?
Other magnetic structures in the solar wind (Sperveslage et al. NPG 7, 191, 2000)

Fig. 2. Examples of magnetic holes in the highest resolution data of the missions Voyager 2 (1.92 s), Helios 1 and 2 (0.25 s); time intervals are three minutes.

Fig. 3. Examples of magnetic enhancements in the highest resolution data of the mission Helios 2; time intervals are one minute except for panel 3.
Evidence of huge magnetic holes in the solar wind
(Stevens & Kasper, JGR 112, A10905 (2007))

(a) This magnetic hole has a cross-sectional width of about 100 proton gyroradii
(b) This hole, however, is about 4000 proton gyroradii across.

Effect of solar wind expansion on mirror modes?

Alternative origin: Alfvén waves propagating at large angle to the ambient magnetic field
[Buti et al., JGR 2001]

Do kinetic effects play a role in big structures?
Main properties of observed structures:

• Structures are quasi-static in the plasma frame

• Small change in the magnetic field direction

• Observed in regions displaying: ion temperature anisotropy $T_{i\perp} > T_{i\parallel}$, \( \beta \) of a few units.

  \textit{(conditions met under the effect of plasma compression in front of the magnetopause).}

  Not always in a mirror unstable regime.

• Magnetic fluctuations mostly affect the parallel component.

• Cigar-like structures, quasi-parallel to the ambient field, with a transverse scale of a few Larmor radii.

• Density is anticorrelated with magnetic field amplitude.

Origin of these structures is still not fully understood.

Usually viewed as nonlinearily saturated states of the mirror instability, or possibly, in particular in the solar wind, remnants of mirror structures created upstream of the point of observation (Winterhalter et al. 1995).

Other recent interpretations:

• trains of slow-mode magnetosonic solitons (Stasiewicz 2004)

• mirror instability is the trigger, generating high amplitude fluctuations that evolve such as to become nonlinear solutions of isotropic or anisotropic plasma equations (Baumgärtel, Sauer & Dubinin 2005)
Linear instability
cold electrons, \( f = f(v_\parallel^2, v_{\perp}) \)

Instability condition:
\[
\Gamma = \beta_\Gamma - \beta_{\perp} - 1 > 0
\]
(Shapiro & Shevchenko 1964)

\[
\beta_{\perp} = \frac{mn}{p_B} \int \frac{v_{\perp}^2}{2} f d^3v \\
\beta_{\parallel} = \frac{mn}{p_B} \int v_{\parallel}^2 f d^3v \\
\beta_\Gamma = -\frac{mn}{p_B} \int \frac{v_{\parallel}^2}{4} \frac{\partial f}{\partial v_{\parallel}^2} d^3v
\]

\( n \) is the background density of the protons, \( m \) their mass
\( p_B = B_0^2 / 8\pi \) background magnetic pressure

Linear growth rate
(near threshold):
\[
\gamma_k = \sqrt{\frac{2}{\pi}} |k_{\parallel}| \bar{v} \left( \Gamma - \frac{3}{2} \bar{v}^2 k_{\perp}^2 - \frac{k_{\parallel}^2}{k_{\perp}^2} \chi \right)
\]

\[
\bar{v}^{-1} = -\frac{\sqrt{2\pi}mn}{p_B} \int \frac{v_{\parallel}^4}{4} \delta(v_{\parallel}) \frac{\partial f}{\partial v_{\parallel}^2} d^3v
\]

\[
\chi = 1 + \frac{1}{2} (\beta_{\perp} - \beta_{\parallel})
\]

\[
\bar{v}^2 = -\frac{mn}{24p_B} \frac{1}{\Omega^2} \int \left( v_{\perp}^6 \frac{\partial f}{\partial v_{\parallel}^2} + 3v_{\perp}^4 f \right) d^3v
\]

For a bi-Maxwellian distribution:
\[
\beta_{\perp} = \frac{m n v_{\text{th} \perp}^2}{p_B}
\]
\[
\beta_{\parallel} = \frac{m n v_{\text{th} \parallel}^2}{p_B}
\]
\[
\bar{v} = \frac{v_{\text{th} \parallel}}{\beta_\Gamma}
\]
\[
\beta_\Gamma = \frac{\beta_{\perp}^2}{\beta_{\parallel}}
\]
\[
\bar{r} = v_{\text{th} \perp} (\beta_\Gamma - \beta_{\perp})^{1/2} / \Omega
\]

Instability condition:
\[
\Gamma^* \equiv \beta_{\perp} \left( \frac{\beta_{\perp}}{\beta_{\parallel}} - 1 \right) - 1 > 0
\]

Growth rate:
\[
\gamma_k = |k_{\parallel}| v_{\text{th} \parallel} \frac{\beta_{\parallel}}{\sqrt{\pi} \beta_{\perp}} \left[ \frac{\beta_{\perp}}{\beta_{\parallel}} - 1 - \frac{1}{\beta_{\perp}} \right]
\]
\[
-\frac{k_{\perp}^2}{k_{\parallel}^2 \beta_{\perp}} \left( 1 + \frac{\beta_{\perp} - \beta_{\parallel}}{2} \right) - \frac{3}{4} \frac{k_{\parallel}^2}{\beta_{\parallel}} \rho_L^2
\]
(\( \rho_L \): ion Larmor radius)
Linear mirror instability (continued)

- Zero-frequency instability.
- Driven by Landau wave-particle resonance and quenched at small-scales by finite Larmor radius effects.

- At least near threshold, it develops at large angle with respect to the ambient field.
  At small or moderate angle and/or smaller $\beta$, Ion Cyclotron Anisotropic Instability can be dominant.

Nevertheless, numerical simulations suggest that mirror modes could dominate in the nonlinear regime ("Competition between the mirror-mode instability and the L-mode electromagnetic ion cyclotron instability: results from comparison of 2-D and 3-D simulations" Shoji, Omura, Tsurutani, Verkhoglyadova http://rp.iszf.irk.ru/hawk/URSI2008/paper/HP04p3.pdf)

See also Hall, J. Plasma Phys. 21, 431 (1979).

Understanding of the nonlinear dynamics is still incomplete.
Magnetic holes are also observed in conditions for which the plasma is linearly stable (BISTABILITY).

Skewness of magnetic fluctuations:
• when negative: magnetic holes
• when positive: magnetic humps

Distance to threshold

Magnetosheath CLUSTER data (Génot et al., AGU 2006)

Soucek, Lucek & Dandouras (JGR 2008): “peaks are typically observed in an unstable plasma, while mirror structures observed deep within the stable region appear almost exclusively as dips”.

Bistability also observed in Jovian magnetosheath (Erdös and Balogh 1996)
Solid blue line: theoretical (bi-Maxwellian) mirror threshold
\[
\frac{T_\perp}{T_\parallel} > 1 + \frac{1}{\beta_\perp}
\]

Dashed-dotted blue line: empirical marginal stability
\[
\frac{T_\perp}{T_\parallel} = 1 + \frac{a}{\beta_\parallel^b} \quad a = 0.83, \quad b = 0.58
\]

Black dashed line: fitted boundary between peaks and dips
\[
\frac{T_\perp}{T_\parallel} = \frac{2.15}{\beta_\parallel^{0.39}}
\]

**Figure 3.** Distribution of mirror modes of different types in the anisotropy-beta plane. Red triangles denote peaks with \( P > 0.3 \), green squares dips (\( P < -0.6 \)) and the remaining ambiguous mirror mode events are marked by grey stars.

Soucek, Lucek & Dandouras, JGR 113, A04203 (2008)

**Solar wind:** “Although the plasma surrounding the holes was generally stable against the mirror instability, there are indications that the holes may have been remnants of mirror mode structures created upstream of the points of observation” (Winterhalter et al. 1995).
Magnetic holes: mostly in subcritical regime

Observations of magnetic holes by Cluster 1 on June, 17th 2002

Magnetic humps: in supercritical regime

Observations of magnetic peaks by Cluster 1 on May, 27th 2005

$C_M < 1$ : subcritical

$C_M > 1$ : supercritical

(for bi-Maxwellian equilibrium)

Observations of mirror structures by Cluster 1 on 15 March 2001

2. Numerical simulations of the Vlasov-Maxwell equations

Shed light on the time evolution and on the origin of the structures.

**Mirror unstable regime near threshold in a large domain**

With a PIC code in a large domain:
Domain size = 2048 c/ωpi
Growth rate: 0.005 Ωp
1024 cells with 500 000 particles/cell

1D simulation:

\[
\theta_{kB} = 72.8^\circ \quad \beta_p \parallel = 1 \quad \beta_p \perp = 1.857 \quad \beta_e = 10^{-2}
\]

A large number of modes are excited.
Humps form and undergo coarsening.
QUESTION: What are the saturation mechanisms of the linear instability?

First mechanism suggested for saturation: based on quasi-linear theory (Shapiro & Shevchenko 1963)

• Assumes space homogeneity (thus absence of coherent structures).

• Can consequently be valid at early times only.

• Requires many modes in interaction, thus an extended domain.

• Mainly associated with a diffusion process in velocity space (dominantly along the ambient field).
**Quasi-linear theory**  
*(Shapiro & Shevchenko 1964)*

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_\parallel} D_{\parallel\parallel} \frac{\partial f}{\partial v_\parallel} + \frac{1}{v_\perp} \frac{\partial}{\partial v_\perp} v_\perp \left( D_{\perp\parallel} \frac{\partial f}{\partial v_\parallel} + D_{\parallel\perp} \frac{\partial f}{\partial v_\perp} \right)
\]

\( f \): velocity distribution function averaged over the space variables

\[
D_{\parallel\parallel} = v_\perp \sum_k \frac{|b_k|^2}{4} \frac{\gamma_k k_\parallel^2}{k_\parallel v_\parallel^2 + \gamma_k^2}
\]

\[
D_{\perp\parallel} = -2 \frac{v_\parallel}{v_\perp} D_{\parallel\parallel}
\]

\[
D_{\parallel\perp} = v_\perp^3 \sum_k \gamma_k b_k^2 / 4
\]

\[
|b_k| = \frac{\delta B_z(k)}{B_0}
\]

\[
\frac{\partial b_k}{\partial t} = \gamma_k b_k
\]

Linear growth rate

**fluctuating magnetic energy**

**distance from threshold**

**maximum growth rate**

*(Hellinger & al., GRL, submitted)*
Perturbation of the space-averaged distribution function

\[ \Delta f = f - f^{(0)} \]

QL theory \[ t = 1.4 \times 10^5 \]

PIC simulation \[ t = 2 \times 10^3 \]

integral over negative values
positive values

\[ v_L/v_A \]

flattening

\[ (f) \]

\[ v_L = 2v_A \]

\[ v_L = 2v_A \]
Quasi-linear theory cannot describe structure formation.
It traces the spatially independent part of the distribution function, while nonlinearities describing space variation (wave-wave interactions) are ignored.

Alternative theory: saturation of mirror modes by relaxation to locally marginal stability
(Kivelson and Southwood 1996, Pantellini 1998).

Phenomenological model where particles are divided in two groups that respond differently to the changing field.

Trapped particles with large pitch angle
Passing particles with small pitch angle

\[ \alpha = \tan^{-1}(v_\perp/v_\parallel) \]

In the rising field regions, trapped particles are excluded by the mirror force, leading to a decrease of the particle pressure (reduction of \( \beta_\perp \)) and evolution to marginal stability (with not important change in the particle energy).

In the well regions, no particle can be excluded. Some trapped particles are cooled by losing perpendicular energy (reduction of the temperature anisotropy). Large reductions in the field are required in the wells in order to cool the trapped population enough to stabilize the system.

This model mostly predicts deep magnetic fields in conditions of marginal stability. It hardly explains the formation of magnetic humps and does not address the phenomenon of bistability.
In a PIC simulation in an extended domain near threshold, the instantaneous distance to threshold reaches negative values, a signature that quasi-linear theory ceases to apply when coherent structures begin to form. The instability continues to take place while $\Gamma < 0$, due to hydrodynamic-type nonlinear effects. Positive skewness: magnetic humps.

No relaxation to marginal stability regime.
PIC hybrid simulations at moderate $\beta$

(Baumgärtel, Sauer & Dubinin, GRL, 2003)

Moderate distance from threshold

Initial random noise in a mirror instable regime leads to the formation of magnetic humps whose number decreases as time elapses.

Magnetic humps form and undergo coarsening.

Figure 4. Space-time evolution of the magnetic field in an uniform, collisionless, anisotropic, mirror-unstable plasma with bi-Maxwellian proton distribution ($\beta_{i\perp} = 5, \beta_{i\parallel} = 2.5$, $\beta_e = 1, \theta = 80^\circ$).
PIC simulation in a small computational domain

Oscillations of the magnetic energy fluctuations with a period consistent with the ion bounce time

\[ \omega_{tr}^2 = \frac{1}{2} v_{th}^2 k_r^2 \left( \frac{\delta B}{B_0} \right) \]

Suggests that particle trapping is at the origin of oscillations.
Saturation by particle trapping in gyrokinetic simulations starting with a single mirror mode (Qu, Lin & Chen, GRL 35, L10108, 2008)

Figure 1. Time history of the amplitude of (a) the perturbed parallel magnetic field and (b) the ion temperature anisotropy for $\beta_{i,\perp} = 2$ and $A_i = 1$.

Anisotropy only weakly reduced

Figure 2. The island formation of the distribution function in the phase-space at $t \cdot \Omega_i = 22.50$ (the cross in Figure 1a). (a) The amplitude of the perturbed parallel magnetic field. (b) The distribution function $f = f_0 + \delta F_\parallel$ in the phase-space. (c and d) The normalized perturbed distribution function $\delta F_\parallel / f_0$ taken for $\mu = 1$ and $\mu = 2$, respectively.
Simulations in a small domain (15x $2\pi c/\omega_{pi}$), using an Eulerian code (no numerical noise)

$\beta || = 15$, $T_\perp/T || = 1.4$ and $\theta = 1.37$

- In a small domain, the quasi-linear phase is not present.
- Amplitude oscillations, associated with particle trapping.

Magnetic hump (and density hole) resulting from the mirror instability, starting from noise.
Magnetic humps form even very close to threshold

\[ \beta_\parallel = 6, \quad \theta = 1.463, \quad T_\perp/T_\parallel = 1.25 \]

**Run 17, By, t=4700**

**Distribution function does not display flattening.**

**No quasi-linear phase.**

**High resolution in velocity space nevertheless required.**

**Time evolution of the unstable modes**

Growth rate of most unstable mode (m=3) : 0.0017 \( \Omega_i \)
3. Modeling the structure formation

A. Asymptotic expansion (near a bi-Maxwellian equilibrium)

Close to threshold, the linearly unstable mirror modes are confined to large scales.

Nonlinear dynamics amenable to a reductive perturbative expansion that isolates mirror modes (Kuznetsov, Passot & Sulem, PRL, 98, 235003, 2007).

At large scales, kinetic effects (Landau damping and finite Larmor radius corrections) are weak and contribute only linearly in the weakly nonlinear regime supposed to develop near threshold.

This argument is validated by a systematic reductive perturbative analysis performed on the Vlasov-Maxwell system (Califano et al. JGR 113, A08212, 2008).

For the sake of simplicity, assume cold electrons with negligible inertia.
Equation governing the proton velocity (derived from Vlasov equation)

\[ \frac{d u_p}{dt} + \frac{1}{\rho_p} \nabla \cdot p_p - \frac{e}{m_p} (E + \frac{1}{c} u_p \times B) = 0 \]

Assuming cold electrons with no inertia:

\[ E = -\frac{1}{c} \left( u_p - \frac{j}{ne} \right) \times B \quad \text{with} \quad j = \left( \frac{c}{4\pi} \right) \nabla \times B \]

\[ p_p = p_\perp n + p_\parallel \tau + \Pi \quad \text{with} \quad n = 1 - \hat{b} \times \hat{b} \quad \tau = \hat{b} \times \hat{b} \quad \hat{b} = B/|B| \]

\[ \rho \frac{d u_p}{dt} = \nabla \left( p_\perp + \frac{|B|^2}{8\pi} \right) + \left( 1 + \frac{4\pi}{|B|^2} (p_\perp - p_\parallel) \right) \frac{(B \cdot \nabla) B}{4\pi} - \hat{b} \frac{|B|^2}{4\pi} (\hat{b} \cdot \nabla) \left( 1 + \frac{4\pi}{|B|^2} (p_\perp - p_\parallel) \right) + \nabla \cdot \Pi \]

In order to address the asymptotic regime, we rescale the independent variables in the form \( X = \sqrt{\varepsilon} x, \ Y = \sqrt{\varepsilon} y, \ Z = \varepsilon z, \ T = \varepsilon^2 t, \) where \( \varepsilon \) measures the distance to threshold, and expand any field \( \varphi \) in the form

\[ \varphi = \sum_{n=0} \varepsilon^{n/2} \varphi_{n/2} \]

Scalings of the space and time variables are suggested by the linear instability growth rate near threshold

\[ \gamma = |k_z| v_{\text{th}} \frac{\beta_\parallel}{\sqrt{\pi} \beta_\perp} \left[ \frac{\beta_\perp}{\beta_\parallel} - 1 - \frac{\beta_\perp}{k_\perp^2 \beta_\parallel} \left( 1 + \frac{\beta_\perp}{\beta_\parallel} \right) - \frac{3}{4 \beta_\perp} k_\perp^2 \rho_L^2 \right] \quad (\rho_L: \text{ion Larmor radius}) \]
In particular, \( B_\perp = \varepsilon^{3/2} B_\perp^{(3/2)} + \varepsilon^{5/2} B_\perp^{(5/2)} + \cdots \) and
\[
B_z = B_0 + \varepsilon B_z^{(1)} + \varepsilon^2 B_z^{(2)} + \cdots.
\]

The perpendicular pressure and the gyroviscous force are to be calculated from the Vlasov equation. The vanishing of the contribution of zeroth order produces the instability threshold.

In this near-threshold asymptotics,
\begin{itemize}
  \item time derivative originates from Landau damping
  \item Landau damping and finite Larmor radius effects arise only linearly
\end{itemize}

\[ E_B = 0 \]

cold electrons without inertia
\[
E_z = \varepsilon^5 E_z^{(5)} + \varepsilon^7 E_z^{(7)} \cdots.
\]

One shows that \( \nabla_\perp \times B_\perp^{(3/2)} = 0 \). By the divergenceless condition:

\[
D_\perp^{(3/2)} = (-\Delta_\perp)^{-1} \nabla_\perp \partial_ Z B_z^{(1)}.
\]

Defining \( b_z = B_z^{(1)} + \varepsilon B_z^{(2)} \) and \( \bar{p}_\perp = \bar{p}_\perp^{(1)} + \varepsilon \bar{p}_\perp^{(2)} \),

the ion-velocity equation reduces to a pressure balance equation

\[
\nabla \left[ \bar{p}_\perp + \frac{B_0}{4\pi} b_z + \frac{\varepsilon}{2} b_z^2 \right] + 2\varepsilon \beta_\perp \left( 1 + \frac{\beta_\perp - \beta_||}{2} \right) \rho_\perp^{(0)} (\Delta_\perp)^{-1} \partial_ Z b_z + \varepsilon \left( \nabla \cdot \Pi \right)_\perp^{(5/2)} = O(\varepsilon^2)
\]

The perpendicular pressure and the gyroviscous force are to be calculated from the Vlasov equation:

For a biMaxwellian equilibrium:

\[
\bar{p}_\perp = \beta_\perp \left( 1 - \frac{\beta_\perp}{\beta_||} \right) \frac{B_0 b_z}{4\pi} + \varepsilon \frac{\sqrt{\pi}}{v_{\text{th}}} \partial_T \left( -\mathcal{H} \partial_ Z \right)^{-1} \frac{\beta_\perp^2}{\beta_||} \frac{B_0 b_z}{4\pi}
\]

\[
- \varepsilon \rho_\perp^{(0)} \left[ \frac{9}{4\beta_\perp} \Delta_\perp b_z \frac{b_z}{B_0} + \left( 1 - 4 \frac{p_\perp}{p_||} + 3 \left( \frac{\beta_\perp}{\beta_||} \right)^2 \left( \frac{b_z}{B_0} \right)^2 \right) \right]
\]

\[
\left( \nabla \cdot \Pi \right)_\perp^{(5/2)} = -\frac{3}{4} \left( 1 - \frac{\beta_\perp}{\beta_||} \right) \overline{p}_\perp^{(0)} r_L^2 \Delta_\perp \nabla_\perp \left( \frac{b_z}{B_0} \right)
\]

\( r_L \): ion Larmor radius

The vanishing of the contribution of zeroth order reproduces the instability threshold.

Dynamical equation obtained at the next order.
Dynamical equation (assuming a bi-Maxwellian equilibrium):

\[
\partial_T \left( \frac{b_z}{B_0} \right) = \frac{v_{th \parallel} \beta_{\parallel}}{\sqrt{\pi} \beta_{\perp}} \left( - \mathcal{H} \partial_Z \right) \left\{ \frac{1}{\varepsilon} \left( \frac{\beta_{\perp}}{\beta_{\parallel}} - 1 - \frac{1}{\beta_{\perp}} \right) \left( \frac{b_z}{B_0} \right) + \frac{3}{4 \beta_{\perp}} r_L^2 \Delta_{\perp} \left( \frac{b_z}{B_0} \right) \right. \\
- \frac{1}{\beta_{\perp}} \left( \frac{\beta_{\perp} - \beta_{\parallel}}{2} \right) \left( \Delta_{\perp} \right)^{-1} \partial_{ZZ} \left( \frac{b_z}{B_0} \right) - \frac{3}{2} \left( \frac{1 + \beta_{\perp}}{\beta_{\perp}^2} \right) \left( \frac{b_z}{B_0} \right)^2 \left. \right\} = O(\varepsilon).
\]

After simple rescaling

\[
\partial_r U = \left( - \mathcal{H} \partial_{\xi} \right) \left[ \sigma U + \Delta_{\perp} U - \Delta_{\perp}^{-1} \partial_{\xi \xi} U - 3U^2 \right]
\]

Here, \(\sigma = \pm 1\), depending on the positive or negative sign of the threshold parameter \(\beta_{\perp}/\beta_{\parallel} - 1 - 1/\beta_{\perp}\).

When the spatial variation are limited to a direction making a fixed angle with the ambient field

\[
\partial_T U = \hat{K}_\Xi \left[ (\sigma + \partial_{\Xi \Xi}) U - 3U^2 \right]
\]

\[\hat{K}_z = -\mathcal{H} \partial_z\]

whose Fourier transform is \(|K_z|\)

where \(\Xi\) is the coordinate along the direction of variation.
Integration above threshold ($\sigma > 1$), with as initial conditions a sine function involving several wavelengths.

After an initial phase of linear instability, formation of a dominant magnetic hole. After a while, solution blows up with a self-similar behavior.

Wave-particle resonance provides the trigger mechanism leading to the linear instability.

Hydrodynamic nonlinearities reinforce the instability, leading to collapse.

Linear FLR effects arrest the linear instability at small scales but cannot cope with hydrodynamic nonlinearities.

At the level of Vlasov-Maxwell eqs, the singularity is the signature of the formation of finite-amplitude structures, through a subcritical bifurcation that cannot be captured perturbatively.

Magnetic holes and not humps are obtained!

When spatial variations are limited to a direction making a fixed angle with the ambient field:

$$\frac{\partial U}{\partial T} = \hat{K}_{\Xi} \left[ \left( \sigma + \frac{\partial^2}{\partial \Xi^2} \right) U - 3U^2 \right]$$

Solution profile near collapse

No stable non-zero stationary solutions to the asymptotic equation

This equation can be written in the form

\[ \frac{\partial U}{\partial T} = -\hat{K}_Z \frac{\delta F}{\delta U} \]

where \( \hat{K}_Z = -\mathcal{H} \partial / \partial Z \) is a positive definite operator (whose Fourier transform is \(|K_Z|\)), and

\[ F = \int \left[ -\frac{\sigma}{2} U^2 + \frac{1}{2} U \Delta^{-1} \frac{\partial^2}{\partial Z^2} U + \frac{1}{2} (\nabla U)^2 + U^3 \right] d^3 R \]

has the meaning of a free energy or a Lyapunov functional. This quantity can only decrease in time, since

\[ \frac{dF}{dT} = \int \frac{\delta F}{\delta U} \frac{\partial U}{\partial T} d^3 R = -\int \frac{\delta F}{\delta U} \hat{K}_Z \frac{\delta F}{\delta U} d^3 R \leq 0. \]

\( dF/dT \) is strictly negative above threshold. It can indeed only vanish for

\[ \frac{\delta F}{\delta U} = \left( \sigma - \Delta^{-1} \frac{\partial^2}{\partial Z^2} + \nabla U \right) U - 3U^2 = 0. \]

Above threshold (\( \sigma=+1 \)): there is no non-zero solutions of \( \star \).
Below threshold (\( \sigma=-1 \)): solutions exist (in the form of magnetic holes).
In 3D, they correspond to saddle points of the free energy and are thus unstable.
In 1D, the solution is the KdV soliton; the linearized operator near this solution has one neutral mode (associated to space translation) with one node and thus a negative energy level: again it is unstable.

(Kuznetsov, Passot & Sulem, JETP Letters, 86, 637, 2007)

- No steady solutions above threshold
- Unstable solutions below threshold
- Blowup of a small-amplitude initial condition above threshold

\[ \text{Subcritical bifurcation} \]

\[ \text{Saturated solution is not amenable to a perturbative calculation} \]
Reductive perturbative expansion performed near bi-Maxwellean equilibrium, retaining only linear kinetic effects, predicts that the nonlinear development of the mirror instability leads to the formation of magnetic holes.

Similar observation when using a more comprehensive semi-fluid description:

**FLR-Landau fluids** (Passot & Sulem, Phys. Plasmas 14, 082502 (2007): Fluid description obtained by closing the moment hierarchy by means of a closure relation aimed to reproduce the linear kinetic theory near a bi-Maxwellian distribution (include linear Landau damping and FLR corrections in the gyrokinetic scaling).

FLR Landau fluids also predict that nonlinear saturation of the mirror instability leads to magnetic holes.
At least one of the two assumptions (bi-Maxwellian equilibrium, linear kinetic effects) is to be challenged.
Extension of the reductive perturbative expansion:

The reductive perturbative expansion near threshold can be extended to any (frozen) smooth equilibrium distribution function $f(v^2_\parallel, v_\perp)$ provided $\tilde{v} > 0$, $\tilde{r}^2 > 0$, and $\chi > 0$.

$$\partial_t b = \sqrt{\frac{2}{\pi}} \frac{\tilde{v}}{1 + 2 \frac{\tilde{v}}{v_\Lambda}} \left( -\mathcal{H} \partial_z \right) \left( \Gamma b + \frac{3}{2} \tilde{r}^2 \Delta_\perp b - \chi \frac{\partial^2}{\Delta_\perp} b - \Lambda b^2 \right)$$

For a bi-Maxwellian distribution, $\beta_\Lambda = 3/2 \beta_\perp^3/\beta_\parallel^2$, thus $\Lambda > 0$ and the model predicts formation of magnetic holes, while humps are observed in the simulations.

This suggests that the early-time QL dynamics affects the forthcoming formation of the structures.

We are thus led to modify eq. (▲) by assuming that the coefficients are not frozen at their initial values but are evaluated from the instantaneous distribution function given by the QL diffusion equation.

For consistency, the contribution of resonant particles are to be retained in the estimate of the nonlinear coupling constant.

$$v_\Lambda^{-1} = \sqrt{2\pi} \frac{mn}{p_B} \int \frac{v_\parallel^6}{8} \delta(v_\parallel) \frac{\partial^2 f}{(\partial v^2_\parallel)^2} d^3 v$$

(▲ ▲)
Results of the simulation of eq. \((\uparrow \uparrow)\)

Formation of magnetic humps
The asymptotic equation cannot capture the saturation of the mirror instability. The asymptotic scaling are broken rather early.

**B. Phenomenological modeling of the saturation**

I. Models based on particle trapping:

*Kivelson and Southwood JGR** **101**, 17365 (1996)
*Pantellini JGR** **103**, 4789 (1998)

Assume a separation of the particle distribution into trapped and untrapped components that respond differently to magnetic field variations. Saturation results from the cooling of trapped particles in magnetic troughs.

Usually predict deep magnetic holes and are hardly consistent with the presence of magnetic humps (only predicted for exceptionally high values of $\beta$); Bi-stability not addressed.

(ii) *Pokhotelov et al. JGR* **113**, A04225 (2008): phenomenological modeling of particle trapping by a prescribing flattening of the parallel distribution function on a range that extends with the strength of the magnetic perturbation. This leads to a renormalization of the time derivative (associated with the quenching of the Landau resonance).

\[
1 - \frac{2}{\pi} \arctan \left( \frac{h^{1/2}}{y} \right) \left( \frac{\partial h}{\partial \tau} \right) = k_{\xi} \left[ 1 + \frac{\partial^2}{\partial \xi^2} \right] (h - h^2)
\]

Prevents wave collapse. The stationary solutions still have the form of KdV solitons. Only holes can result from this approach.
II. Effect of variation of the local ion Larmor radius:
when phenomenologically supplemented to the asymptotic equation, it makes the model consistent with Vlasov-Maxwell simulations.

Motivation:

In regions of weaker magnetic field (and/or large $T_\perp$), ion Larmor radius is larger, making the stabilizing effects of finite radius corrections more efficient than in the linear regime. Consequently, mirror instability is more easily quenched in magnetic field minima than in maxima, making magnetic humps more likely to form in the saturating phase of the mirror instability.

Assume a bi-Maxwellian equilibrium

$$\partial_t U = (-\mathcal{H}\partial_\xi) \left[ \sigma U + \Delta_\perp U - \Delta_\perp^{-1} \partial_\xi \xi U - 3U^2 \right]$$

$$\Delta_\perp U \rightarrow \frac{1}{1 + \alpha U} \Delta_\perp U + \frac{4}{9 (1 + \alpha U)^2} \Delta_\perp^2 U$$

$v$ (taken equal to 0.01) is related to the size of the box

$$\alpha = \frac{2\beta_\perp}{1 + \beta_\perp} \left[ \beta_\perp \left( \frac{T_\perp}{T_\parallel} - 1 \right) - 1 \right]$$

Using conservation of magnetic moment,

$$\rho_L^2 \propto T_\perp / |B|^2 \propto 1 / |B| \approx 1 / B_z$$

Furthermore, in addition to Laplacian which results from the leading order expansion of a nonlocal operator associated with FLR corrections, we also retain the next order contribution.
Evolution after saturation of linear instability

Coarsening of magnetic humps resulting from the mirror instability in the framework of the phenomenological model.

Magnetic holes predicted by the phenomenological model initiated by a random noise of small amplitude when $\sigma > +1$ and of large amplitude when $\sigma < -1$. 

$\sigma = 1.54$

$\sigma = 0.5$

$\sigma = -0.05$

$\sigma = -0.3$

$\sigma = -0.4$
Skewness of magnetic fluctuations in the quasi-stationary regime

**Figure 10.** Variation of the skewness with the parameter $\sigma \alpha$, as predicted by the phenomenological model.

I.C.: small-amplitude random noise in supercritical regime
large-amplitude random noise in subcritical regime

Cluster data: statistic of structures observed in the magnetosheath.

4. Formation of magnetic holes when starting with large initial perturbations

Subcritical solutions (i.e. below threshold)

Model simulation

Vlasov simulation in a small domain

FIG. 4. Quasistatic solution of the saturated equation for $\sigma = -1$, $\nu = 0.01$, and $\alpha = 0.32$, obtained with large initial perturbations.

\[ \beta_\parallel = 6, \quad \theta = 1.463, \quad T_\perp = T_\parallel \]

Large-amplitude magnetic holes survive even far below threshold.

Magnetic humps do not survive
PIC hybrid simulations at moderate $\beta$ below threshold (Baumgärtel, Sauer & Dubinin, GRL, 2003)

A localized magnetic perturbation in the form of a finite-amplitude hole persists

Figure 1. Space-time evolution of a magnetic depression in an otherwise uniform, collisionless, isotropic high-$\beta$ plasma with $\beta_e = 2.5$, $\beta_p = 1$, $\theta = 80^\circ$. The initial perturbation is prescribed as $\delta B_y/B_0 = -0.5 \exp \left(-x^2/h^2\right)$ with $h = 10 \cdot c/\omega_{pi}$; $\delta B_y$ is set to zero at $t = 0$.

A localized magnetic perturbation in the form of a finite-amplitude hump relaxes

Figure 3. Space-time evolution of a magnetic compression $\delta B_y/B_0 = 0.5 \exp \left(-x^2/h^2\right)$, other parameters the same as in Figure 1.
Eulerian Vlasov simulations in a small domain for **large-amplitude initial perturbations above threshold**

Magnetic hole (and density hump), starting with a large amplitude magnetic field depression, above threshold.

Domain size: $15 \times 2\pi \frac{c}{\omega_{pi}}$, with $\beta_\parallel=6,$ $T_\perp/T_\parallel=1.2$ and $\theta=1.463$
Large-amplitude magnetic holes are found to be stable solutions even far above threshold.

\[ \beta_\parallel = 6, \quad \theta = 1.463, \quad T_\perp/T_\parallel = 1.5 \]
Mirror structures are different from soliton solutions:

Magnetic field component perpendicular to the plane \((\mathbf{k}, \mathbf{B}_o)\) is symmetric with respect to the center of the magnetic hole.

Differently, soliton models based on anisotropic Hall-MHD (Stasiewicz 2004, Mjolhus 2006) predict an antisymmetric \(b_y\) profile.

Stasiewicz, JGR 110, A03220 (2005)
Formation of **magnetic holes** from small-amplitude noise in a mirror unstable plasma

PIC simulation **far from threshold** starting from random noise.

Early-formed humps transform into holes.

1024 cells with 500 000 particles/cell; Domain size=1024 c/ω_{pi}

Growth rate: 0.156 Ω⁻¹

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Distance to threshold remains slightly positive. The system is continuously stirred and coarsening is less efficient. In particular, there are no isolated structures.

θ_{kB} = 50.5° (most unstable angle)

β_p|| = 1 \hspace{1cm} β_{p⊥} = 4

β_e = 10^{-2}

**Formation of magnetic holes** from small-amplitude noise in a mirror unstable plasma

Distance to threshold remains slightly positive. The system is continuously stirred and coarsening is less efficient. In particular, there are no isolated structures.

**Late transition from magnetic humps to magnetic holes**

β is decreasing, which favours nonlinear stability of magnetic holes.

No such transition at larger β (e.g., β_p|| = 2).

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**Form of biMaxwellian**

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**Instantaneous skewness becomes negative**
The distribution function remains close to bi-Maxwellian.
No flattening of the distribution function.
5. Summary

• Numerical integrations in a large domain of VM equations demonstrate the existence of an early phase described by quasi-linear theory, followed by a regime where coherent structures form.

• In a small domain, no quasi-linear phase but significant oscillations due to particle trapping.

• The structures resulting from the saturation of the mirror instability are magnetic humps.

• Stable solutions in the form of large-amplitude magnetic holes exist both above and below threshold.

• Holes can also form in the late evolution of an extended system when initialized far from threshold.

• Reductive perturbative expansion of VM eqs near threshold leads to an equation with a finite-time singularity, signature of a subcritical bifurcation. Nature of the structures depends on the equilibrium distribution function. An early QL phase can provide the conditions for the formation of magnetic humps.

• A phenomenological modeling retaining ion Larmor radius variations predicts the formation of saturated magnetic humps above threshold and existence of stable large-amplitude magnetic holes mainly below threshold, in agreement with observations and numerical simulations.

Hellinger, Kuznetsov, Passot, Sulem & Travnicek, GRL, 34, xxxx, DOI:10.1029/2008GL036805. in press.
6. Open questions

- Would it be possible to retain nonlinear FLR effects and the influence of trapped particles within an asymptotic theory?

- Are there conditions where mirror instability saturates by quasi-linear effects?

- Can one observe, very close to threshold, the signature of the singularity predicted by the reductive perturbative expansion?

- How to understand quantitatively the transition from humps to holes observed in large-box simulations far from threshold?

- Perform Vlasov-Maxwell simulations in two or three dimensions.

- What is the role of the mirror structures on the magnetopause boundary? Can they trigger micro-reconnection events?